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The Differential Invariants of Space.

By J. E. Wright.

The object of this paper is to solve the three following problems:

(I) The determination of the differential invariants of all orders, of space of any number of dimensions.

(II) The determination of invariants of any manifold in this space under any transformation which leaves its "shape" unaltered.

(III) The determination of the "deformation" invariants of any manifold in this space.

In the statement of the problems, the word "invariant" is used to include the whole class of Gaussian Invariants, Parameters and Covariants. The problems are considered solved when a method is given for determining a complete functionally independent set of invariants by direct processes.

In problem (I) for example, it will be shown that all the invariants may be expressed as the algebraic invariants of certain forms, and a method will be given for writing down these forms in succession. At one stage of the work it is found simpler to discard certain of these forms and to introduce instead a complete set of invariants due to them.

In (II) the solutions are given as the invariants of algebraic forms, with the exception of those corresponding to the set introduced in (I).

The parameters which arise in the solution of (III) are also expressed in terms of algebraic invariants, but for the determination of the Gaussian invariants, and the covariants, it is found simpler to make use of a method whereby they are expressed in a different manner. The method in question leads to the complete set of invariants in every case, but the expression of the solutions as algebraic invariants has many advantages from some points of view. The chief advantage is perhaps that the well known parameters are immediately recognized.

(III) is really equivalent to the determination of the invariants of a quadratic differential form. There is indeed an apparent limitation on the generality of the form, but it is readily seen that the generality of the results is not thereby affected, and that in fact the problem of the determination of a functionally independent complete set of invariants for a quadratic form is also solved. Lie* suggested this problem for the differential form in two variables. Using Lie's method Żorawski † investigated the invariants of the first and second orders for a general quadratic form. He also considered the question of the number of functionally independent invariants of any order. This question of number has engaged the attention of several investigators, but the complete results are given by C. N. Haskins.‡ His method does not, however, lead to the expressions for the invariants themselves. Forsyth || has obtained the invariants of the first, second, and third orders, of space of three dimensions. His method is, however, unsuitable for the determination of invariants of higher orders. He has also § obtained the invariants of the first, second, and third orders which are of the type sought in (II) for a surface in space of three dimensions.

Maschke,** by the use of symbolic methods has developed a process for determining invariants of any order from known invariants. But his work does not suggest any obvious method of determining a complete set.

§1. The method pursued in this paper for the solution of problem (I) consists essentially in a factorisation of the problem into two others, each of which can be readily solved. Take any system of variables $u_1, u_2 \ldots\ldots u_m$, and let them denote a system of curvilinear coördinates in ordinary space of m dimensions. If the square of the linear element be

$$ds^2 \equiv \sum_{r=1}^{m} \sum_{s=1}^{m} a_{rs} du_r du_s,$$

where a_{rs} is a function of the u's, it must be possible to find $x_1, x_2 \ldots\ldots x_m$, functions of the u's, such that

$$ds^2 = dx_1^2 + dx_2^2 + \ldots\ldots dx_m^2.$$

* Ueber Differentialinvarianten. Math. Ann., Vol. XXIV (1884), pp. 574-575.

† Ueber Biegungsinvarianten, Acta Mathematica, Vol. XVI (1892-93), pp. 1-64.

‡ Trans. Amer. Math. Soc., Vol. III (1902), pp. 71-91; also Trans. Amer. Math. Soc., Vol. V (1904), pp. 167-192.

|| Philosophical Transactions, Series A, Vol. 202 (1903), pp. 277-333.

§ Philosophical Transactions, series A, Vol. 201 (1903), pp. 329-402.

** Trans. Amer. Math. Soc., Vol. I (1900), pp. 197-204; and Vol. IV (1903), pp. 445-469.

It is easy to prove that if one set of functions $x_1 \ldots . x_m$ is given, the most general possible set is given by performing a general orthogonal transformation and a translation on the set x.

In fact if $x_1, x_2 \ldots . x_m$ and $x'_1, x'_2 \ldots . x'_m$ are two different sets, there must exist the relations

$$x'_i = \beta_i + \sum_{j=1}^{m} \alpha_{ij} x_j, \qquad (i = 1, 2 \ldots . m)$$

where the α's and β's are constants such that

$$\sum_{k=1}^{m} \alpha_{ik} \alpha_{jk} = 0 \qquad (i \neq j)$$

$$\sum_{k=1}^{m} \alpha_{ik}^2 = 1 .$$

Now any invariant will be a function of the a's and their derivatives. It will, therefore, be a function of the x's and their derivatives. This function must be invariant under the most general orthogonal transformation and under the most general translation.

It is now easy to see that the problem considered separates into two parts:

(A) The determination of all invariants under a general transformation on the u's, and subject to the condition that the x's are invariant.

(B) The selection from these of those functions which are invariantive when a translation and an orthogonal transformation are performed on the x's.

The first of these is equivalent to the determination of all the invariants of any number of functions of the set of variables $u_1, u_2 \ldots . u_m$. Let these functions be $f^{(1)}(u_1, u_2 \ldots . u_m), f^{(2)}(u_1, u_2 \ldots . u_m) \ldots . f^{(r)}(u_1, u_2 \ldots . u_m)$. We take as the variables occurring in the invariants

(1) all possible derivatives of the f's.

(2) $u_1, u_2 \ldots . u_m$.

(3) $du_1, du_2 \ldots . du_m, d^2u_1, d^2u_2 \ldots . d^2u_m$, etc.

It is true, as pointed out by Forsyth,* that we may take account of the ratios of the set of variables (3), by introducing equations of the type $\phi(u_1 u_2 \ldots . u_m) = 0$. It is, however, simpler to preserve these variables.

* Philosophical Transactions, series A, Vol. 201 (1903), p. 333.

There is one simplification which may be made ; there is no need to take account of more than m functions f, for any other function may be expressed in terms of the first m of them. Hence any derivative of this function may be expressed in terms of the functions $f^{(1)}, f^{(2)} \ldots . f^{(m)}$ and their derivatives. It, therefore, follows that any invariant involving this function can be expressed in terms of $f^{(1)}, f^{(2)}, \ldots . f^{(m)}$ and their derivatives. The invariant is, therefore, reducible.

To solve the problem (A), we make use of Lie's* method, with slight modifications in the details of the work. The method is the following: Let F be any invariant involving the variables specified. On this we perform the most general transformation of the group of point transformations of the variables u, and assume that it becomes F'. Then the condition for invariance is $F' = \Omega^{\mu} F$, where Ω is the Jacobian of the transformation, and μ is a number.

Let $\frac{dF}{dt}$ denote the effect on F of the most general infinitesimal operator of the group, then Ω^{μ} becomes $1 + \mu \left(\sum_{r=1}^{m} \frac{\partial \xi_r}{\partial u_r} \right) \delta t$, where $\frac{du_r}{dt} = \xi_r \, (r = 1, 2, \ldots m)$,

and
$$\frac{dF}{dt} = \mu \left\{ \sum_{r=1}^{m} \frac{\partial \xi_r}{\partial u_r} \right\} F. \tag{1}$$

If F satisfies this equation for all possible operators, then it is an invariant of the type desired. Now since the group is the most general in the variables u, the ξ's are arbitrary functions of their arguments, and hence F must be a function satisfying the system of equations obtained by equating to zero the coefficients in (1) of the various derivatives of the ξ's. As proved by Lie (l.c.) the system of equations thus obtained is complete, and therefore the number of functionally independent solutions is $M - N$, where M is the number of variables, and N the number of linearly independent equations.

To determine the equation system we require the increments under the infinitesimal transformation of the variables in F. The detailed expressions for these increments are not needed for the present work, but Forsyth's† method for their determination is preferable to Lie's, and is therefore used.

Let u_i denote the original, and u'_i the transformed variables, and let $u_i + k_i$ become $u'_i + k'_i$, $(i = 1, 2 \ldots . m)$.

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* Loc. cit., pp. 564-566.

† Philosophical Transactions, series A, Vol. 201 (1903) pp. 336-340.

Then $k_i' = u_i' + k_i' - u_i'$,

$$= u_i + k_i + \xi_i (u + k) \, \delta t - [u_i + \xi_i (u) \, \delta t]$$

where $\xi_i (u)$ is written for $\xi_i (u_1 \, u_2 \, \ldots . \, u_m)$.

Hence $k_i' = k_i + [\xi_i (u + k) - \xi_i (u)] \, \delta t$, and therefore if ϕ denotes any function of the variables u, and ϕ' its transformed,

$$\phi (u + k) = \phi' (u' + k') = \phi' (u' + k + \overline{\xi (u + k) - \xi (u) \, \delta t})$$

$$= \phi' (u' + k) + \sum_{i=1}^{m} [\xi_i (u + k) - \xi_i (u)] \frac{\partial \phi' (u' + k)}{\partial (u_i' + k_i)} \, \delta t + \ldots .$$ or, if small

quantities of order higher than the first are neglected

$$-\frac{d\phi (u + k)}{dt} = \sum_{i=1}^{m} [\xi_i (u + k) - \xi_i (u)] \frac{\partial \phi (u + k)}{\partial (u_i + k_i)},$$ where $\frac{d}{dt}$ operates only

on u, and not on k. This equation holds for all values of the variables k, and therefore the coefficients of corresponding powers of k on both sides may be equated. In this way are obtained the increments of all the derivatives of the functions f. For our present purpose we only require those terms in the expression for any increment which involve the highest derivatives of the ξ's.

Let $f_{\alpha_1 \alpha_2 \ldots \alpha_m}$ denote $\dfrac{\partial^n f}{\partial u_1^{\alpha_1} \partial u_2^{\alpha_2} \ldots . \partial u_m^{\alpha_m}}$, where $\alpha_1 + \alpha_2 + \ldots . + \alpha_m = n$.

Then

$$-\frac{d}{dt} f_{\alpha_1 \alpha_2 \ldots . \alpha_m} = \sum_{i=1}^{m} (\xi_i)_{\alpha_1 \alpha_2 \ldots . \alpha_m} \frac{\partial f}{\partial u_i}$$ + terms involving lower

derivatives of the ξ's. We write $u' \; u'' \ldots . \; u^{(\lambda)} \ldots .$ for the variables du, $d^2 u, \ldots . \; d^{(\lambda)} u, \ldots .$ The increments of these variables may be written down immediately. In fact

$$\frac{d}{dt} \, u_j^{(\lambda)} = \left\{ \sum_{i=1}^{m} u_i' \frac{\partial}{\partial u_i} \right\}^{\lambda} \xi_j$$ + terms involving lower

derivatives of the ξ's. The expression $\left\{ \sum u_i' \dfrac{\partial}{\partial u_i} \right\}^{\lambda}$ is supposed expanded by the multinomial theorem, and then applied as an operator to ξ_j.

It is convenient to classify invariants according to 'order'. An invariant is of the n^{th} order when the increments of its variables involve n^{th} derivatives of the ξ's but no higher derivatives. We call two invariants of the n^{th} order 'independent' when one cannot be expressed in terms of the other together with invariants of order less than n. Suppose that the invariants of orders up to $n-1$ are known. Then the invariants of orders up to n consist of these and a certain number of independent invariants of the n^{th} order. Let F be an invariant of the n^{th} order. It must satisfy the system of equations obtained by equating to zero the coefficients of the n^{th} derivatives of the ξ's in equation (1),

$$\sum_{k=1}^{m} \frac{\partial f^{(k)}}{\partial u_i} \frac{\partial F}{\partial f^{(k)}_{\alpha_1 \alpha_2 \ldots\ldots \alpha_m}} + \frac{n!}{\alpha_1!\,\alpha_2!\ldots\ldots\alpha_m!} (u_1')^{\alpha_1} (u_2')^{\alpha_2} \ldots\ldots (u_m')^{\alpha_m} \frac{\partial F}{\partial u_i^{(n)}} = 0 . \qquad (2)$$

where $i = 1, 2 \ldots\ldots m$, and $\alpha_1\ \alpha_2\ \ldots\ldots\ \alpha_m$ take all positive integral and zero values subject to the condition $\alpha_1 + \alpha_2 + \ldots\ldots \alpha_m = n$.

In addition, F must satisfy the equations obtained from the lower derivatives of the ξ's. It is easy to see that m independent solutions of the equations (2) are $d^n f^{(1)}, d^n f^{(2)}, \ldots\ldots d^n f^{(m)}$, where $d^n f$ denotes the n^{th} total differential of f. Also the number of derivatives of F in these equations is obviously greater by m than the number of equations. Hence we have all the functionally independent solutions in variables of the n^{th} order, provided the equations are linearly independent. One of the determinants of the matrix of the equations may be shown to be

$$\left\{ J \begin{pmatrix} f^{(1)} & f^{(2)} & \ldots\ldots & f^{(m)} \\ u_1 & u_2 & \ldots\ldots & u_m \end{pmatrix} \right\}^{\frac{(m+r-1)!}{r!\,(m-1)!}} .$$

The functions f are assumed independent, and therefore their Jacobian does not vanish. Hence the determinant mentioned does not vanish, and therefore the equations are linearly independent. Hence all the solutions of the equations (2) are obtained. It may readily be verified that each of the solutions $d^n f$ satisfies the remaining equations, and is an absolute invariant; this is also obvious from the form of the solutions. We may now use these solutions to get rid of n^{th} order variables from the remaining equations. When this is done the equations become precisely those for invariants of orders less than or equal to $(n-1)$. Continuing this process we finally arrive at the complete system of invariants,

$$d^{\lambda} f^{(\mu)}, \qquad \begin{pmatrix} \lambda = 2, 3 \ldots\ldots n \\ \mu = 1, 2 \ldots\ldots m \end{pmatrix},$$

together with the solutions of the system of equations for invariants of the first order. A slight modification is here necessary, owing to the fact that F occurs explicitly. It is easy to see that we have the m solutions $df^{(1)}\ df^{(2)}\ \ldots.\ df^{(m)}$, which are absolute invariants, and there yet remains one other integral, which is manifestly the Jacobian of the f's. This is a relative invariant. In fact, if the transformed variables are $U_1,\ U_2\ \ldots.\ U_m$,

$$J\begin{pmatrix} f^{(1)} & \ldots. & f^{(m)} \\ U_1 & \ldots. & U_m \end{pmatrix} \times J\begin{pmatrix} U_1 & \ldots. & U_m \\ u_1 & \ldots. & u_m \end{pmatrix} = J\begin{pmatrix} f^{(1)}, & f^{(2)} & \ldots. & f^{(m)} \\ u_1, & u_2 & \ldots. & u_m \end{pmatrix}$$

and therefore the μ of equation (1) is -1.

Hence we have the theorem: *A complete functionally independent system of invariants of m functions f in the variables $u_1,\ u_2,\ \ldots.\ u_m$, involving variables up to the n^{th} order, is given by*

$$\begin{array}{llll} d\,f^{(1)}, & d\,f^{(2)}, & \ldots. & d\,f^{(m)}, \\ d^2f^{(1)}, & d^2f^{(2)}, & \ldots. & d^2f^{(m)}, \\ \vdots & & & \vdots \\ d^nf^{(1)}, & d^nf^{(2)}, & \ldots. & d^nf^{m}, \end{array}$$

which are absolute invariants, together with

$$J\begin{pmatrix} f^{(1)}, & f^{(2)}, & \ldots. & f^{(m)} \\ u_1, & u_2, & \ldots. & u_m \end{pmatrix},$$

which is a relative invariant with $\mu = -1$.

The most general invariant is therefore a function of the f's and the above absolute invariants, multiplied by some power of J.

§2. We are now in a position to determine all the invariants of ordinary space of m dimensions.

We call the functions $f^{(1)}, f^{(2)},\ \ldots.\ f^{(m)},\ x_1,\ x_2,\ \ldots.\ x_m$. The invariants must be functions of

$$J,$$

$$\begin{array}{llll} x_1, & x_2, & \ldots. & x_m, \\ dx_1, & dx_2, & \ldots. & dx_m, \\ d^2x_1, & d^2x_2, & \ldots. & d^2x_m, \\ \vdots & & & \vdots \\ d^nx_1, & d^nx_2, & \ldots. & d^nx_m, \end{array}$$

and of any number of functions $f^{(k)}(x_1, x_2, \ldots x_m)$, and of their derivatives. They must be invariants under the most general translation, and under the most general orthogonal transformation.

As in the previous case, there is no need to consider more than m functions f, since $f^{(s)}$, $(s > m)$, can be expressed in terms of $f^{(1)}, f^{(2)} \ldots f^{(m)}$. We may at once take account of the translation; it is equivalent to the condition that the x's do not occur explicitly. For the orthogonal transformation, the transformation scheme for the x's is

$$x_i' = x_i + \left\{ \sum_{k=1}^{m} \alpha_{ik} x_k \right\} \delta t,$$

where $\alpha_{ik} = -\alpha_{ki}$ for all values of i and k.

We pursue the same method as before to determine the invariants. There is now, however, the important limitation that the second and higher derivatives of the increments of the x's vanish.

Let the increment of x_i be denoted by ξ_i, then $\frac{\partial \xi_i}{\partial x_i}$ is zero, and therefore the fundamental equation (1) becomes

$$\frac{dF}{dt} = 0.$$

The number of operators in $\frac{d}{dt}$ is $\frac{1}{2} m (m-1)$, since each operator corresponds to an independent constant α. Therefore, provided the operators are unconnected, the number of solutions is $\frac{1}{2} m (m-1)$ less than the number of variables. Now taking account of differentials and of differential coefficients as far as the n^{th} order, and assuming m functions f, the number of variables is

$$1 \text{ of type } J,$$

$$mn \text{ of type } d^r x_s,$$

$$m \left\{ \frac{(m+n)!}{m!\, n!} - 1 \right\} \text{ of type } \frac{\partial^\lambda f}{\partial x_1^{\beta_1} \partial x_2^{\beta_2} \ldots \partial x_m^{\beta_m}}.$$

Altogether there are therefore

$$\frac{(m+n)!}{(m-1)!\, n!} + (n-1) m + 1 \text{ variables,}$$

and therefore the number of solutions is

$$\frac{(m+n)!}{(m-1)!\; n!} + (n-1)\, m + 1 - \tfrac{1}{2}\, m\,(m-1)\,,$$

provided the equations for the invariants are independent.

In order to obtain the equations, the increments of the variables are required. For the derivatives of the f's use is made of the equation

$$-\frac{d}{dt} f(x+k) = \sum_{i=1}^{m} \left[\xi_i(x+k) - \xi_i(x)\right] \frac{\partial f(x+k)}{\partial (x_i + k_i)}\,.$$

In this case

$$\xi_i(x+k) - \xi_i(x) = \sum_{j=1}^{m} k_j \alpha_{ij}\,,$$

and therefore

$$-\frac{d}{dt} \frac{\partial^n f}{\partial x_1^{r_1}, \partial x_2^{r_2} \ldots \partial x_m^{r_m}} = \sum_{i=1}^{m} \sum_{j=1}^{m} \alpha_{ij}\, r_j \frac{\partial^n f}{\partial x_1^{s_1}, \partial x_2^{s_2} \ldots \partial x_m^{s}}$$

$$\left(\begin{matrix} s_i = r_i + 1\,;\; s_j = r_j - 1\,;\; s_\lambda = r_\lambda,\, \lambda \neq i,\, \lambda \neq j\,; \\ r_1 + r_2 + \ldots r_m = n \end{matrix} \right).$$

Now consider the algebraic form of the n^{th} order in the variables $X_1, X_2, \ldots X_m$,

$$A_n \equiv \left\{ \sum_{i=1}^{m} X_i \frac{\partial}{\partial x_i} \right\}^n f(x_1, x_2 \ldots x_m)\,,$$

and let the orthogonal transformation already used be performed on the variables X. The increments of the coefficients are precisely those given above. In addition we notice that the increments for the magnitudes $d^r x$ are exactly similar to those for the magnitudes x, and that J is an absolute invariant.

We therefore immediately obtain the general result:

The functionally independent set of invariants of orders up to and including the n^{th}, of space of m dimensions, are J, and the orthogonal algebraic invariants of the system of m-ary forms

$$\begin{matrix} A_1^{(1)}, & A_1^{(2)}, & \ldots & A_1^{(m)}, \\ A_2^{(1)}, & A_2^{(2)}, & \ldots & A_2^{(m)}, \\ \vdots & & & \vdots \\ A_n^{(1)}, & A_n^{(2)}, & \ldots & A_n^{(m)}, \end{matrix}$$

$$\sum_{i=1}^{m} d^r x_i\, X_i, \qquad (r = 1, 2 \ldots n).$$

The most general invariant of the space under a general point transformation on the variables u, is therefore a function of the functions f, and of the algebraic invariants given above, multiplied by some power of J.

It is worthy of note that the algebraic forms A, are the polar forms of the functions J. The linear forms

$$\sum_{i=1}^{m} d^r x_i \, X_i$$

may be excluded, provided there are m functions f, if certain additional invariants are taken account of. In fact, it is clear that the form

$$\sum_{i=1}^{m} d^r x_i \, X_i$$

leads to the functionally independent set of invariants

$$\sum_{i=1}^{m} d^r x_i \, \frac{\partial f^{(\rho)}}{\partial x_i}, (\rho = 1, 2 \, \ldots . \, m)$$

and it follows at once that there is no need to retain the linear forms specified, provided we add to the set of invariants the expressions

$$d^\lambda f^{(\rho)} . \qquad \begin{pmatrix} \lambda = 1, 2 \, \ldots . \, n \\ \rho = 1, 2 \, \ldots . \, m \end{pmatrix}$$

which are obviously functionally independent invariants.

It is convenient to modify the result obtained by including the quadratic form

$$\sum_{i=1}^{m} X_i^2 ,$$

and then the most general invariant is seen to be a function of

$$\begin{array}{llll} d f^{(1)}, & d f^{(2)}, & \ldots . & d f^{(m)}, \\ d^2 f^{(1)}, & & \ldots . & d^2 f^{(m)}, \\ \vdots & & & \vdots \\ d^n f^{(1)}, & & \ldots . & d^n f^{(m)}, \end{array}$$

and of the general algebraic invariants of the forms A, and

$$\sum_{i=1}^{m} X_i^2 ,$$

multiplied by some power of J.

We now transform from the variables X to new variables U given by the scheme

$$X_i = \sum_{j=1}^{m} \frac{\partial x_i}{\partial u_j} U_j, \qquad (i = 1, 2 \ldots m).$$

The Jacobian of this transformation is J, and the discriminant of the quadratic form ΣX_i^2 changes from 1 to J^2. Hence the most general invariant is a function of the quantities $d^\lambda f$, and of the general algebraic invariants of the A's and ΣX_i^2, expressed in the new variables.

Let the form

$$\sum_{i=1}^{m} X_i^2 \text{ become } \sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij} U_i U_j,$$

then it will be shown that the coefficients of the algebraic forms may be expressed in terms of the a's, the f's and their derivatives with respect to the variables u. We proceed to calculate these forms in the new variables.

$$A_1 \text{ becomes } \sum_{i=1}^{m} U_i \frac{\partial f}{\partial u_i}.$$

In the general case

$$A_p^{(\rho)} = \sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{k=1}^{m} \cdot\cdot \sum_{\lambda=1}^{m} \sum_{\mu=1}^{m} \sum_{\nu=1}^{m} \cdot\cdot \frac{\partial x_i}{\partial u_\lambda} \frac{\partial x_j}{\partial u_\mu} \frac{\partial x_k}{\partial u_\nu} \cdot\cdot \frac{\partial^p f^{(\rho)}}{\partial x_i \partial x_j \partial x_k \cdot\cdot} U_\lambda U_\mu U_\nu \cdot\cdot,$$

where there are n letters $i, j, k \ldots$, and also n letters $\lambda, \mu, \nu, \ldots$. We use

$$\frac{p!}{\lambda!\, \mu!\, \nu! \ldots}\, {}_pB_{\lambda, \mu, \nu, \ldots}$$

to denote the coefficient of U_λ, U_μ, U_ν .... in the above expression for A_p.

It is convenient to introduce at this stage the well known three index symbols of Christoffel

$$\begin{bmatrix} \lambda, \mu \\ \nu \end{bmatrix} \equiv \tfrac{1}{2} \left\{ \frac{\partial a_{\lambda\nu}}{\partial u_\mu} + \frac{\partial a_{\mu\nu}}{\partial u_\lambda} - \frac{\partial a_{\lambda\mu}}{\partial u_\nu} \right\}$$

and we have the relations

$$\sum_{i=1}^{m} \frac{\partial^2 x_i}{\partial u_j\, \partial u_k} \frac{\partial x_i}{\partial u_\lambda} = \begin{bmatrix} j, k \\ \lambda \end{bmatrix}.$$

Now $\frac{\partial}{\partial u_\rho}\, {}_qB_{\lambda,\mu,\nu,\ldots} = {}_{q+1}B_{\rho,\lambda,\mu,\nu,\ldots}$

$$+ \sum_{ijk\ldots}^{m} \frac{\partial^q f}{\partial x_i\, \partial x_j\, \partial x_k \ldots} \cdot \left\{ \sum \frac{\partial^2 x_i}{\partial u_\rho\, \partial u_\lambda} \frac{\partial x_j}{\partial u_\mu} \frac{\partial x_k}{\partial u_\nu} \cdots \right\}$$

where the second Σ denotes that there is a term corresponding to each of the letters λ, μ, ν, ... and that these are added.

Also
$$ {}_qB_{p,\mu,\nu,\ldots} = \sum_{i,j,k,\ldots}^{m} \frac{\partial^q f}{\partial x_i\, \partial x_j\, \partial x_k \ldots} \frac{\partial x_i}{\partial u_p} \frac{\partial x_j}{\partial u_\mu} \frac{\partial x_k}{\partial u_\nu} \cdots . $$

We solve the m equations obtained by giving p the values 1, 2, ... m, for the quantities

$$\sum_{j,k,\ldots} \frac{\partial^q f}{\partial x_i\, \partial x_j\, \partial x_k \ldots} \frac{\partial x_j}{\partial u_\mu} \frac{\partial x_k}{\partial u_\nu} \cdots ,$$

and substitute the result in the above equation. Hence

$$\frac{\partial}{\partial u_\rho}\, {}_qB_{\lambda,\mu,\nu,\ldots} = {}_{q+1}B_{\rho,\lambda,\mu,\nu,\ldots}$$

$$+ \sum \left\{ \sum_{i=1}^{m} \sum_{p=1}^{m} {}_qB_{p,\mu,\nu,\ldots}\, M_p^i \frac{\partial^2 x_i}{\partial u_\rho\, \partial u_\lambda} \right\} \frac{1}{J},$$

where M_p^i is the cofactor of $\frac{\partial x_i}{\partial u_p}$ in J. This equation may be written

$$ {}_{q+1}B_{\rho,\lambda,\mu,\nu,\ldots} - \frac{\partial}{\partial u_\rho}\, {}_qB_{\lambda,\mu,\nu,\ldots} =$$

$$\Sigma \frac{1}{J} \begin{vmatrix} 0 & \frac{\partial^2 x_1}{\partial u_\rho \partial u_\lambda}, & \frac{\partial^2 x_2}{\partial u_\rho \partial u_\lambda}, & \cdots & \frac{\partial^2 x_m}{\partial u_\rho \partial u_\lambda} \\ {}_qB_{1,\mu,\nu,\ldots} & \frac{\partial x_1}{\partial u_1}, & \frac{\partial x_2}{\partial u_1}, & \cdots & \frac{\partial x_m}{\partial u_1} \\ {}_qB_{2,\mu,\nu,\ldots} & \frac{\partial x_1}{\partial u_2}, & \frac{\partial x_2}{\partial u_2}, & \cdots & \frac{\partial x_m}{\partial u_2} \\ \vdots & & & & \vdots \\ {}_qB_{m,\mu,\nu,\ldots} & \frac{\partial u_m}{\partial x_m}, & \cdots & \cdots & \frac{\partial x_m}{\partial u_m} \end{vmatrix}$$

We multiply the determinant on the right by J, and observe that $J^2 = \Delta$, where Δ is the discriminant of the quadratic form

$$\sum_{r, s,} a_{rs}\, U_r\, U_s\,.$$

We thus obtain the result

$$_{q+1}B_{\rho, \lambda, \mu, \nu, \ldots} - \frac{\partial}{\partial u_\rho}\, {}_qB_{\lambda, \mu, \nu, \ldots} =$$

$$\Sigma \begin{vmatrix} 0 & , \begin{bmatrix} \rho, \lambda \\ 1 \end{bmatrix}, & \begin{bmatrix} \rho, \lambda \\ 2 \end{bmatrix}, & \ldots & \begin{bmatrix} \rho, \lambda \\ m \end{bmatrix} \\ {}_qB_{1, \mu, \nu, \ldots} & a_{11} & a_{12} & \ldots & a_{1m} \\ {}_qB_{2, \mu, \nu, \ldots} & a_{21} & a_{22} & \ldots & a_{2m} \\ \vdots & & & & \vdots \\ {}_qB_{m, \mu, \nu, \ldots} & a_{m1} & a_{m2} & \ldots & a_{mm} \end{vmatrix} \frac{1}{\Delta}$$

Where the summation contains a term for each letter $\lambda, \mu, \nu, \ldots$.

Multiply both sides of this equation by $U_\rho\, U_\lambda\, U_\mu\, U_\nu \ldots$ and sum for all values of $\rho, \lambda, \mu, \nu, \ldots$. Also let the operator

$$\sum_{\rho=1}^{m} U_\rho \frac{\partial}{\partial u_\rho}, \text{ be denoted by } \Omega.$$

Then $S_{q+1} - \Omega\, S_q$

$$= \frac{1}{\Delta} \begin{vmatrix} 0, & F_1 & , F_2 & , \ldots & F_m \\ \frac{\partial S_q}{\partial U_1}, & a_{11} & , a_{12} & , \ldots & a_{1m} \\ \vdots & & & & \vdots \\ \frac{\partial S_q}{\partial U_m}, & a_{m1} & , a_{m2}, & \ldots & a_{mm} \end{vmatrix} \quad \ldots \quad (3)$$

Where S_q denotes the form previously called A_q, and F_p is the quadratic form

$$\sum_{\rho=1}^{m} \sum_{\lambda=1}^{m} \begin{bmatrix} \rho, \lambda \\ p \end{bmatrix} U_\rho U_\lambda.$$

It thus appears that the coefficients of S_{q+1} involve the a's, their derivatives, the coefficients of the forms S_q and their derivatives. But S_1 is obviously Ωf, and therefore the coefficients of S_q involve the a's, their derivatives of orders up

to $q-1$, and the derivatives of the f's of orders up to q. These forms may be readily calculated in succession by means of the given equation.

The final result may now be stated. Let $u_1, u_2 \ldots\ldots u_m$ be any coördinates in space of m dimensions, and let the element of length ds be given by the equation

$$ds^2 = \sum_{r=1}^{m} \sum_{s=1}^{m} a_{rs}\, du_r\, du_s .$$

A complete functionally independent system of differential invariants of orders up to and including n is given by the expressions

$$d^{\lambda} f^{(\rho)} \qquad (\lambda = 1, 2, \ldots . n;\ \rho = 1, 2, \ldots . m)$$

together with the algebraic invariants of certain m-ary forms. These forms consist of:

(I) The quadratic $$\sum_{r=1}^{m} \sum_{s=1}^{m} a_{rs}\, U_r\, U_s .$$

(II) n forms of orders 1, 2, ... n, corresponding to each function $f(u_1, u_2, \ldots . u_m)$.

The coefficients of the forms (II) involve the derivatives of the f's, the a's and their derivatives. Equation (3) enables these forms to be written down in succession.*

§3. We next consider any manifold of dimensions r in the m dimensional space. In this case there are two types of invariants. There is, in the first place, a class of invariants corresponding to transformations which preserve the 'shape' of the manifold, that is to say do not alter distances apart of points on the manifold, the distances being measured through the m dimensional space. These contain a subclass of deformation invariants, namely functions which remain unchanged when the manifold is subjected to a transformation which merely preserves lengths measured in the manifold itself. Let the manifold in question be given by $u_{r+\lambda} = 0\ (\lambda = 1 \ldots . m - r)$.

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* Forsyth, in his memoir already quoted, "*The Differential Invariants of Space*" (p. 320), gives three of the forms S, namely, those for a function ϕ when $m = 3$ and $n = 3$.

Then the first class of invariants is included in the general space set. We may now, however, take as $m - r$ of the functions f the variables $u_{r+\lambda}$. In addition $du_{r+\lambda} = 0 \; \ldots . \; d^n u_{r+\lambda} = 0$.

It is quite easy to see that the result of the last paragraph may be modified by omitting the invariants $df^{(1)} \ldots . df^{(m)}$, and taking $du_1, du_2, \ldots . du_m$, as the variables in the algebraic forms. Then the set of forms includes the $m - r$ linear ones $du_{r+\lambda}$ $(\lambda = 1, 2, \ldots . m - r)$ and therefore the invariants of the manifold are the invariants and covariants of the forms in r variables obtained by putting $du_{r+\lambda} = 0$ in the m-ary set,* and in addition there are the invariants

$$d^p f^{(\rho)} \; (p = 2, \ldots . n; \; \rho = 1, 2 \ldots . m).$$

In all the coefficients of the forms the $u_{r+\lambda}$'s $(\lambda = 1 \ldots . m - r)$ are finally put equal to zero.

There are two types of forms to be calculated, namely those belonging to a function $u_{r+\lambda}$, and those belonging to a function $\phi\,(u_1, u_2, \ldots . u_r)$. We make the assumption that the variables $u_{r+\lambda}$ may be so selected that in the expression

$$ds^2 = \sum_{\alpha\beta}^{m} a_{\alpha\beta}\, du_\alpha\, du_\beta$$

the coefficient $a_{\lambda\mu} = 0$ if $\lambda > r$, $\mu \leq r$. With this selection of variables

$$S_2^{(r+\lambda)} \text{ becomes } \frac{M_\lambda}{M_0},$$

where M_i is the cofactor of b_i in

$$\begin{vmatrix} b_0 & F_{r+1,} & \ldots . & F_m \\ b_1 & a_{r+1,r+1} & \ldots & a_{r+1,m} \\ \vdots & & & \vdots \\ b_{m-r} & a_{r+1,m} & \ldots & a_{m,m} \end{vmatrix}$$

The coefficients of these forms obviously cannot be expressed in terms of the magnitudes $a_{\alpha\beta}$ $(\alpha, \beta = 1, 2 \ldots . r)$. Let the coefficients in question be $L_{ij}^{(r+\lambda}$ $(ij = 1 \ldots . r)$.

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* See Grace and Young. Algebra of Invariants (1903), p. 266.

Then $S_{q+1}^{(r+\lambda)} - \Omega_1 S_q^{(r+\lambda)}$

$$= \frac{1}{\Delta_1} \begin{vmatrix} 0 & F_1 & F_2 & \cdots & F_r \\ \dfrac{\partial S_q^{(r+\lambda)}}{\partial U_1} & a_{11} & a_{12} & \cdots & a_{1r} \\ \vdots & & & & \vdots \\ \dfrac{\partial S_q^{(r+\lambda)}}{\partial U_r} & a_{r1} & & \cdots & a_{rr} \end{vmatrix}$$

where Δ_1 is the discriminant of

$$\sum_{\alpha\beta}^{r} a_{\alpha\beta} U_\alpha U_\beta, \qquad \Omega_1 \equiv \sum_{=1}^{r} U_j \frac{\partial}{\partial u_j},$$

and the variables U_j of the forms are du_j $(j = 1, \ldots r)$. Now the coefficients of the forms $F_1, F_2, \ldots F_r$ are the three index symbols $\begin{bmatrix} ij \\ k \end{bmatrix}$ $(i, j, k = 1, 2 \ldots r)$ and they are therefore expressible in terms of the derivatives of the $a_{\alpha\beta}$'s where $\alpha, \beta \not> r$.

It therefore follows that the coefficients of the forms $S^{(r+\lambda)}$ $(\lambda = 1, 2 \ldots m - r)$ may all be expressed in terms of the coefficients of ds^2 for the manifold, their derivatives, and the derivatives of the coefficients of the forms $S_2^{(r+\lambda)}$.

The coefficients of the forms $S_q^{(r+\lambda)}$ are seen to be the generalization of the "Fundamental Magnitudes"* of order q of a surface in space of three dimensions.

The forms corresponding to a function $\phi\,(u_1, u_2, \ldots u_r)$ are exactly similar to the general forms for space, for $S_1 = \Omega_1 \phi$.

$$S_{q+1} = \Omega_1 S_q + \frac{1}{\Delta_1} \begin{vmatrix} 0 & F_1 & \cdots & F_r \\ \dfrac{\partial S_q}{\partial U_1} & a_{11} & \cdots & a_{1r} \\ \vdots & & & \vdots \\ \dfrac{\partial S_q}{\partial U_r} & a_{r1} & \cdots & a_{rr} \end{vmatrix}.$$

* See Forsyth, Messenger of Mathematics, Vol. 32, 1903, pp. 68 et seq.

We may now get rid of covariants of the forms in question by taking account of the invariants $d\phi_1 \ldots d\phi_r$, and the final solution of problem II is obtained. The general statement is the following: Let there be any manifold in space of m dimensions, and let its linear element be given by the equation

$$ds^2 = \sum_{\alpha,\beta=1}^{r} a_{\alpha\beta}\, du_\alpha\, du_\beta .$$

In addition let there be r functions $\phi\,(u_1 \ldots u_r)$. Then a complete system of algebraically independent differential invariants of the manifold is given by

$$d^\lambda\,\phi^{(\sigma)} \qquad (\lambda = 1, 2 \ldots n;\ \sigma = 1, 2 \ldots r)$$

together with the algebraically independent invariants of a system of r-ary forms. These forms are:

(1) A quadratic $\Sigma a_{\alpha\beta} du_\alpha du_\beta$.

(2) $m - r$ quadratics whose coefficients are the fundamental magnitudes of the second order for the manifold.

(3) $m - r$ forms of order q ($q = 3, \ldots n$) of which the coefficients are the fundamental magnitudes of order q, and which may be expressed in terms of the a's, their derivatives, and the fundamental magnitudes of the second order and their derivatives.

(4) A set of n forms of orders $1, 2 \ldots n$ corresponding to each function ϕ. The coefficients are functions of the a's, the ϕ's, and their derivatives. Forsyth* gives a particular case of this solution, namely that for $n = 3$, $m = 3$, $r = 2$. He, however, assumes at the beginning that there exist certain invariant differential forms corresponding to the sets (2) and (3) whereas in our case these forms have arisen naturally in the course of the development of the method.

§4. Problem (III) still remains for consideration. In the first place, it is clear that $d^\lambda\,\phi^{(\sigma)}$ ($\lambda = 1, 2 \ldots n$; $\sigma = 1, 2 \ldots r$) and the algebraic invariants of the sets (1) and (4) of algebraic forms given in the preceding section are deformation invariants of the quadratic differential form ds^2.†

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* Philosophical Transactions, Ser. A, Vol. 201 (1903), p. 357.

† There is one condition to which the differential form considered is subject. It is assumed possible to express it as the sum of the squares of m perfect differentials. This condition involves no limitation on the form, for it may be proved that any form can be so expressed, provided $m \gtrless \frac{1}{2} r\,(r+1)$. See Goursat, Leçons sur l'intégration des Équations aux Dérivées Partielles du premier ordre (1891), p. 11.

In addition to these there are others due to the sets (2) and (3), for it is easy to see that there are relations among the fundamental magnitudes of the second order, the a's and their derivatives,* and hence some of the invariants due to these sets must be expressible in terms of the derivatives of the a's alone. The solution of problem (I) shows that these are all zero in that particular case. It is interesting in this connection, to note the six quantities $\Theta_1, \Theta_2, \ldots . \Theta_6$ obtained by Forsyth † and originally due to Cayley. ‡ The method hitherto pursued leads us no further in the determination of "Gaussian Invariants," and we must have recourse to another. The problem to be solved is the determination of the invariants of a quadratic form, when there are no associated functions ϕ. Use is made of Lie's general method by the introduction into the group hitherto used of a certain number of new variables which are invariant to the group. ‖ Let the quadratic form be

$$\sum_{h=1}^{r} \sum_{k=1}^{r} a_{hk}\, du_h\, du_k ,$$

and introduce invariant variables $\alpha_1, \alpha_2, \ldots . \alpha_r$. The a's are functions of the u's, and the u's are taken to be functions of the α's, which are a set of independent variables. We now seek for invariants, under the most general transformation on the u's, which involve as variables

(1) the u's and their derivatives with respect to the α's,
(2) the a's and their derivatives with respect to the u's,
subject to the condition that the quadratic form is invariant.

It is easy to see

(a) that $\sum_{h=1}^{r} \sum_{k=1}^{r} a_{hk} \frac{\partial u_h}{\partial \alpha_\lambda} \frac{\partial u_k}{\partial \alpha_\mu}$ $(\lambda, \mu = 1, 2, \ldots . r)$ is an absolute invariant,

(b) that if H is any absolute invariant so also is $\frac{\partial H}{\partial \alpha}$.

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* In the case of a surface in space of three dimensions the Gaussian curvature leads to such a relation.

† Philosophical Transactions, Series A, Vol. 202 (1903), p. 306.

‡ Collected Mathematical Papers, Vol. XII, pp. 12, 13.

‖ Lie, Math. Ann., Vol. XXIV (1884), p. 564, lines 22 et seq.

These two statements (a) and (b) are sufficient to give the clue to the complete set of invariants sought.

Let $T_{\lambda\mu}$ denote the expression (a), and form all possible derivatives up to and including the order $n - 1$, of the T's with respect to the α's. These are a complete functionally independent set of invariants. It is easy to prove the independence of this set, for if α_λ is taken to be u_λ, $(\lambda = 1, 2 \ldots . r)$ the T's and their derivatives become the a's and their derivatives, and the a's are arbitrary functions of their arguments. Hence the number of functionally independent invariants obtained is

$$r \frac{(r+1)}{2} \frac{(n+r-1)!}{(r-1)!\, n!}$$

Now the set of equations for these invariants obtained by the Lie process is precisely that considered by C. N. Haskins,* with certain additional terms due to the presence of derivatives of the u's with respect to the a's. He shows that if $n > 3$ the equations are independent, and therefore it is certain that if $n > 3$ our equations are independent. Now they possess altogether as variables

$$r \frac{(r+1)}{2} \frac{(n+r-1)!}{(r-1)!\, n!} \; a\text{'s and their derivatives,}$$

$$r \left\{ \frac{(n+r)!}{(r-1)!\, (n+1)!} - 1 \right\} \text{ derivatives of the } u\text{'s.}$$

Also the equations are in number

$$r \left\{ \frac{(n+r)!}{(r-1)!\, (n+1)!} - 1 \right\},$$

and therefore they possess exactly

$$\frac{r(r+1)}{2} \cdot \frac{(n+r-1)!}{(r-1)!\, n!}$$

functionally independent common solutions. But this number has been obtained, and therefore the system of equations has been completely solved. Any invariant is therefore a function of the T's and their derivatives, and the Gaussian Invari-

* Trans. Amer. Math. Soc., Vol. 3 (1902), pp. 74-76.

ants are the particular functions which do not involve the derivatives of the u's with respect to the α's. These may be obtained by processes of elimination, and hence the problem of the determination of the Gaussian Invariants of a quadratic form is completely solved.

It is clear that the introduction of any functions $\phi\ (u_1, u_2, \ldots\ u_r)$ leads to additional invariants $\frac{d\phi}{d\alpha}$ etc., and therefore the parameters may be obtained in this way. The extension to any number of forms of any degree is immediate, but this question is reserved for future discussion.

BRYN MAWR COLLEGE, PENNA., *April*, 1905.

ON DIFFERENTIAL INVARIANTS*

BY

JOSEPH EDMUND WRIGHT

Introduction.

In the consideration of differential equations there arise expressions, such as for instance, the JACOBI-POISSON alternant of two first order partial equations, which are in their nature invariantive to all contact transformations. An important problem immediately presents itself, namely, the obtaining of all invariants of this type, that is of all invariants, with respect to contact transformations, of differential expressions.

In this paper are obtained all such invariants of a restricted type.

The restrictions are the following:—

(1) The only expressions considered are: (a) expressions of the first order with m dependent and n independent variables; (b) expressions of the second order with one dependent variable.

(2) The invariants are only of the first order, that is to say, they involve only first derivatives of the differential expressions.

The variables assumed to occur in case (a) are

$$x_1, \cdots, x_n \text{ (the independent variables)},$$

$$z_1, \cdots, z_m \text{ (the dependent variables)},$$

$$p_k^i \equiv \frac{\partial z_i}{\partial x_k} \qquad (i=1, \cdots, m;\ k=1, \cdots, n),$$

$$p_{kl}^i \equiv \frac{\partial^2 z_i}{\partial x_k \partial x_l} \qquad (i=1, \cdots, m;\ k, l=1, \cdots, n),$$

and

$$\frac{\partial f_\lambda}{\partial x_k}, \quad \frac{\partial f_\lambda}{\partial z_i}, \quad \frac{\partial f_\lambda}{\partial p_k^i} \qquad (\lambda=1, \cdots, r;\ i=1, \cdots, m;\ k=1, \cdots, n),$$

where $f_\lambda(x, z, p_k^i)$, $(\lambda = 1, \cdots, r)$, are the differential expressions considered.

In addition the variables dx_k, dz_i, dp_k^i will be assumed to enter, subject to the conditions

$$dz_i = \sum_{k=1}^{n} p_k^i dx_k \qquad (i=1, \cdots, m),$$

$$dp_k^i = \sum_{l=1}^{n} p_{kl}^i dx_l \qquad (i=1, \cdots, m;\ k=1, \cdots, n).$$

* Presented to the Society April 29, 1905. Received for publication November 12, 1904.

In invariants corresponding to case (b) the variables are the same, with the exception that m is 1, and that there are additional variables

$$\frac{\partial f_\lambda}{\partial p_{kl}} \qquad (\lambda = 1, \cdots, r;\, k,\, l = 1, \cdots, n),$$

where $f_\lambda(x, z, p_k, p_{kl})$ are the differential expressions considered.

The method used is a modification of that given by LIE in his paper "Ueber Differentialinvarianten,"* in which invariants for certain simpler types of infinitesimal transformations are obtained. In this paper LIE shows (p. 566) that by the method there outlined, a series of invariants may be obtained satisfying a system of linear differential equations which form a complete system.

In the paper mentioned LIE suggests a problem connected with invariants of surfaces, the class desired being that which does not change owing to "deformation" of the surface.

ŻORAWSKI † attempted the solution of this question and found a class of such invariants.

FORSYTH, ‡ in 1903 attacked the more general question of invariants due to a purely arbitrary point transformation performed on the surface, and also obtained in this manner the differential invariants of space. In his paper certain modifications are made on the LIE-ŻORAWSKI method, one modification being that he sought for relative, as well as absolute, invariants. The method as modified by FORSYTH will be used here.

The invariant sought will, therefore, be such that if F denote its expression in the original, F_1 in the transformed variables, we shall have

$$F_1 = \Omega F,$$

where Ω is a function depending only on the transformation.

Now if the transformation were a general one in the variables considered, it is well known that Ω would be some power of the Jacobian of the transformation. But the transformation is not perfectly general.

In fact, in our most general case, the Jacobian of X_k, Z_i, P^i_k, P^i_{kl} with reference to x_k, z_i, p^i_k, p^i_{kl}, where capitals denote transformed variables, and where $\{k, l = 1, 2, \cdots n;\, i = 1, 2 \cdots m\}$ breaks up into two factors, the first of which is

$$J_1 \equiv J\left(\frac{X_1 X_2 \cdots X_n Z_1 \cdots Z_m P'_1 P'_2 \cdots P^m_n}{x_1 x_2 \cdots p^m_n}\right),$$

and the second

$$J_2 \equiv J\cdot\left(\frac{\cdots P^i_{kl} \cdots}{\cdots p^i_{kl} \cdots}\right).$$

*LIE, Mathematische Annalen, vol. 24 (1884), pp. 537–578.

† ŻORAWSKI, Acta Mathematica, vol. 16 (1892–3), pp. 1–64.

‡ FORSYTH, Philosophical Transactions, ser. A., vol. 201 (1903) pp. 329–402, and ser. A., vol. 202 (1904), pp. 277–333.

Further, if the transformation is a point transformation, J_1 breaks up into two factors,

$$J_0 \equiv J\left(\frac{X_1 X_2 \cdots X_n Z_1 \cdots Z_m}{x_1 x_2 \cdots x_n z_1 \cdots z_m}\right),$$

$$J_1 \equiv J\left(\frac{P'_1 P'_2 \cdots P^m_n}{p'_1 p'_2 \cdots p^m_n}\right).$$

Now if the number of dependent variables is greater than one, it may easily be shown that the most general contact transformation possible is an extended point transformation.

The discussion will be limited to those cases in which the factor Ω is of the form

$$J_0^{\mu_0} J_1^{\mu_1} J_2^{\mu_2} \cdots$$

when the number of dependent variables is greater than one, and

$$\bar{J}_1^{\mu_1} J_2^{\mu_2} \cdots$$

when there is only one dependent variable.

Now let F be an invariant of the type considered, and let an infinitesimal contact transformation be performed on F.

The condition for invariance is that, t being the parameter of the transformation,

$$F_1 = \Omega F, \qquad \text{or} \qquad \frac{dF}{dt} = F\frac{d\Omega}{dt}$$

when $d\phi/dt$ denotes the increment of ϕ due to the infinitesimal transformation and F_1 is F in the tranformed variables. Expressing the fact that this equation holds for all such transformations as considered, we obtain a complete system of linear differential equations, the solutions of which are the invariants desired.

In the course of the work the values of certain increments are required, and they will be given now, before we consider the various cases in detail.

The following notation is used throughout:—

$$\frac{dx_k}{dt} \equiv \xi_k, \qquad \frac{dz_i}{dt} \equiv \zeta_i, \qquad \frac{dp^i_k}{dt} \equiv \pi^i_k, \qquad \frac{dp^i_{kl}}{dt} \equiv \pi^i_{kl}, \qquad \theta_i \equiv \zeta_i - \sum_{k=1}^{n} p^i_k \xi_k,$$

$$\frac{\partial f}{\partial x_k} \equiv X_k, \qquad \frac{\partial f}{\partial z_i} \equiv Z_i,$$

$$\frac{\partial f}{\partial p^i_k} \equiv P^i_k, \qquad \frac{\partial f}{\partial p^i_{kl}} \equiv P^i_{kl},$$

where f is used to denote one of the forms considered. If it is desired to specify any one of the f's particularly, the notation

$$f_\lambda, \qquad X_{\lambda,k}, \qquad P^i_{\lambda,k}, \qquad P^i_{\lambda,kl},$$

etc., is used.

S denotes a summation taken over all the expressions f.

In addition,

$$dx_k \equiv a_k, \qquad dz_i \equiv b_i, \qquad dp^i_k \equiv c^i_k, \qquad dp^i_{kl} \equiv c^i_{kl},$$

$$\frac{d}{dx_k} \equiv \frac{\partial}{\partial x_k} + \sum_{\lambda=1}^{m} p^\lambda_k \frac{\partial}{\partial z_\lambda} + \sum_{\lambda,\, l} p^\lambda_{kl} \frac{\partial}{\partial p^\lambda_l}.$$

Using this notation we have the following increments:

$$\frac{dJ_0}{dt} \equiv \sum_{i=1}^{m} \frac{\partial \theta_i}{\partial z_i} + \sum_{k=1}^{n} \frac{d\xi_k}{dx_k},$$

$$\frac{dJ_1}{dt} \equiv n \sum_{i=1}^{m} \frac{\partial \theta_i}{\partial z_i} - m \sum_{k=1}^{n} \frac{d\xi_k}{dx_k},$$

$$\frac{dJ_2}{dt} \equiv \tfrac{1}{2} n(n+1) \sum_{i=1}^{m} \frac{\partial \theta_i}{\partial z_i} - m(n+1) \sum_{k=1}^{n} \frac{d\xi_k}{dx_k}$$

when m is greater than unity, and

$$\frac{d\bar{J}_1}{dt} \equiv (n+1) \frac{\partial \theta}{\partial z},$$

$$\frac{dJ_2}{dt} \equiv (n+1) \sum_{\lambda=1}^{n} \frac{d}{dx_h} \left(\frac{\partial \theta}{\partial p_h} \right) + \tfrac{1}{2} n(n+1) \frac{\partial \theta}{\partial z}$$

when m is unity.

The quantities π^i_k, π^i_{hk}, etc., are readily obtained in terms of the ξ's and ζ's and their derivatives by the method given in LIE-ENGEL, *Theorie der Transformationsgruppen*, vol. 1, p. 544, *et seq.*

In the case, however, where m is unity the increment of the variables in an extended infinitesimal contact transformation may be expressed in a particularly simple manner.

The theorem is as follows:

Let there be an extended infinitesimal contact transformation in the n independent variables $x_1, x_2, \cdots, x_n$ and the dependent variable z, and let $p_{hkl\cdots}$ denote $\partial^r z/\partial x_h \partial x_k \partial x_l \cdots$, where there are r letters $h, k, l, \cdots$. Then the increment of $p_{hkl\cdots}$ due to the transformation is $(d^r \theta / dx_h dx_k dx_l \cdots)\delta t$, where θ, with the usual notation, is equal to

$$\zeta - \sum_{l=1}^{n} p_l \xi_l,$$

and $(d^r \theta / dx_h dx_k dx_l \cdots)$ denotes a total differentiation of θ in which the terms containing the highest derivatives of z are omitted.

For example, in the case when there are two independent variables,

$$\pi_1 = \frac{\partial \theta}{\partial x_1} + p_1 \frac{\partial \theta}{\partial z}, \qquad \pi_2 = \frac{\partial \theta}{\partial x_2} + p_2 \frac{\partial \theta}{\partial z},$$

$$\pi_{11} = \frac{\partial^2 \theta}{\partial x_1^2} + 2p_1 \frac{\partial^2 \theta}{\partial z\, \partial x_1} + p_1^2 \frac{\partial^2 \theta}{\partial z^2}$$

$$+ 2p_{11}\left(\frac{\partial^2 \theta}{\partial x_1\, \partial p_1} + p_1 \frac{\partial^2 \theta}{\partial z\, \partial p_1}\right) + 2p_{12}\left(\frac{\partial^2 \theta}{\partial x_1\, \partial p_2} + p_1 \frac{\partial^2 \theta}{\partial z\, \partial p_2}\right)$$

$$+ p_{11}^2 \frac{\partial^2 \theta}{\partial p_1^2} + 2p_{11}p_{12} \frac{\partial^2 \theta}{\partial p_1 \partial p_2} + p_{12}^2 \frac{\partial^2 \theta}{\partial p_2^2} + p_{11} \frac{\partial \theta}{\partial z}, \text{ etc.}$$

This theorem is known to be true * for the increments of the first derivatives of z, and it may be easily proved for higher derivatives by induction.

The increments of the quantities $X_{\lambda,k}$, $Z_{\lambda,i}$, $P^i_{\lambda,k}$, $P^i_{\lambda,hk}$, owing to the infinitesimal contact transformation are determined by the method given by FORSYTH in his paper already quoted.†

Using this method, we have the results

$$-\frac{dX_\lambda}{dt} = \sum_{k=1}^{n} \frac{\partial \xi_k}{\partial x_\lambda} X_k + \sum_{i=1}^{m} \frac{\partial \zeta_i}{\partial x_\lambda} Z_i + \sum_{ik} \frac{\partial \pi^i_k}{\partial x_\lambda} P^i_k + \sum_{ihk} \frac{\partial \pi^i_{hk}}{\partial x_\lambda} P^i_{hk},$$

$$-\frac{dZ_\lambda}{dt} = \sum_{k=1}^{n} \frac{\partial \xi_k}{\partial z_\lambda} X_k + \sum_{i=1}^{m} \frac{\partial \zeta_i}{\partial z_\lambda} Z_i + \sum_{ik} \frac{\partial \pi^i_k}{\partial z_\lambda} P^i_k + \sum_{ihk} \frac{\partial \pi^i_{hk}}{\partial z_\lambda} P^i_{hk},$$

$$-\frac{dP^\lambda_\mu}{dt} = \sum_{k=1}^{n} \frac{\partial \xi_k}{\partial p^\lambda_\mu} X_k + \sum_{i=1}^{m} \frac{\partial \zeta_i}{\partial p^\lambda_\mu} Z_i + \sum_{ik} \frac{\partial \pi^i_k}{\partial p^\lambda_\mu} P^i_k + \sum_{ihk} \frac{\partial \pi^i_{hk}}{\partial p^\lambda_\mu} P^i_{hk},$$

$$-\frac{dP^\lambda_{\mu\nu}}{dt} = \sum_{k=1}^{n} \frac{\partial \xi_k}{\partial p^\lambda_{\mu\nu}} X_k + \sum_{i=1}^{m} \frac{\partial \zeta_i}{\partial p^\lambda_{\mu\nu}} Z_i + \sum_{ik} \frac{\partial \pi^i_k}{\partial p^\lambda_{\mu\nu}} P^i_k + \sum_{ihk} \frac{\partial \pi^i_{hk}}{\partial p^\lambda_{\mu\nu}} P^i_{hk}.$$

The increments of the quantities a_k, b_i, c^i_k, etc., are readily calculated, for the transformation changes x_k into $x_k + \xi_k \delta t$, and therefore dx becomes $dx_k + d\xi_k\, \delta t$.

Hence

$$\frac{da_k}{dt} = \sum_{l=1}^{n} \frac{\partial \xi_k}{\partial x_l} a_l + \sum_{\lambda=1}^{m} \frac{\partial \xi_k}{\partial z_\lambda} b_\lambda + \sum_{l,\lambda} \frac{\partial \xi_k}{\partial p^\lambda_l} c^\lambda_l.$$

Similarly

$$\frac{db_i}{dt} = \sum_{l=1}^{n} \frac{\partial \zeta_i}{\partial x_l} a_l + \sum_{\lambda=1}^{m} \frac{\partial \zeta_i}{\partial z_\lambda} b_\lambda + \sum_{l,\lambda} \frac{\partial \zeta_i}{\partial p^\lambda_l} c^\lambda_l,$$

with similar expressions for the other increments of this type.

* See LIE-ENGEL, *Theorie der Transformationsgruppen*, vol. 2, p. 82, 252.

† FORSYTH, Philosophical Transactions, vol. 201, p. 337, 338.

§ 1.

We shall now proceed to the determination of invariants of expressions of the first order. The case in which there is only one dependent variable differs from the others in that the most general infinitesimal contact transformation is not an extended point transformation. This case will therefore be considered independently of the other.

Assuming F to be an invariant, and performing on it the infinitesimal transformation corresponding to a function

$$\theta \equiv \zeta - \sum_{k=1}^{n} p_k \xi_k,$$

we have the equation

$$\frac{dF}{dt} + \mu(n+1)\theta_z F = 0.$$

If the variables occurring in F are

$$x_1, \cdots, x_n, z, p_1, \cdots, p_n, X_{\lambda,1}, \cdots, X_{\lambda,n}, Z_\lambda, P_{\lambda,1}, \cdots, P_{\lambda,n} \quad (\lambda = 1 \cdots r),$$

this equation becomes on expansion,

$$\begin{aligned}
\mu(n+1)\theta_z F &+ \left(\theta - \sum_{i=1}^{n} p_i\theta_{p_i}\right)\frac{\partial F}{\partial z} + \sum_{i=1}^{n}(\theta_{x_i} + p_i\theta_z)\frac{\partial F}{\partial p_i} - \sum_{i=1}^{n}\theta_{p_i}\frac{\partial F}{\partial_{x_i}} \\
&- \mathbf{S}\sum_{i=1}^{n}\frac{\partial F}{\partial X_i}\left\{\sum_{\rho=1}^{n} X_\rho(-\theta_{p_\rho x_i}) + \left(\theta_{x_i} - \sum_{\rho=1}^{n} p_\rho\theta_{p_\rho x_i}\right)Z + \sum_{\rho=1}^{n}(\theta_{x_\rho x_i} + p_\rho\theta_{x_i z})P_\rho\right\} \\
&- \mathbf{S}\frac{\partial F}{\partial Z}\left\{\sum_{\rho=1}^{n} X_\rho(-\theta_{p_\rho z}) + \left(\theta_z - \sum_{\rho=1}^{n} p_\rho\theta_{p_\rho z}\right)Z + \sum_{\rho=1}^{n}(\theta_{x_\rho z} + p_\rho\theta_{zz})P_\rho\right\} \\
&- \mathbf{S}\sum_{i=1}^{n}\frac{\partial F}{\partial P_i}\left\{\sum_{\rho=1}^{n} X_\rho(-\theta_{p_\rho p_i}) + \sum_{\rho=1}^{n}(-p_\rho\theta_{p_\rho p_i})Z\right. \\
&\left. + \sum_{\rho=1}^{n}(\theta_{p_i x_\rho} + p_\rho\theta_{z p_i})P_\rho + \theta_z P_i\right\} = 0.
\end{aligned}$$

Now θ is a perfectly arbitrary function of the variables $x_1 \ldots x_n, z, p_1 \cdots p_n$ and the above equation must be satisfied for all values of θ.

Hence we may equate to zero the coefficients of the derivatives of θ, and thus obtain the system of linear partial equations which F must satisfy.

The system is the following:

From θ

$$(1) \qquad F_z = 0.$$

From θ_{x_i},

$$(2) \qquad F_{p_i} - \mathbf{S}ZF_{X_i} = 0 \qquad (i = 1 \cdots n).$$

From θ_{p_i},

(3) $$F_{x_i} = 0 \qquad (i=1, \cdots, n).$$

From θ_z,

(4) $$\mu(n+1)F + \sum_{i=1}^{n} p_i F_{p_i} - \mathbf{S} Z F_Z - \mathbf{S} \sum_{i=1}^{n} P_i F_{P_i} = 0.$$

From $\theta_{x_i x_k}$,

(5) $$\mathbf{S} P_k F_{X_i} + \mathbf{S} P_i F_{X_k} = 0 \qquad (i=1, \cdots, n;\ k=1, \cdots, n).$$

From $\theta_{x_i z}$,

(6) $$\mathbf{S} \sum_{\rho=1}^{n} p_\rho P_\rho F_{X_i} + \mathbf{S} P_i F_Z = 0 \qquad (i=1, \cdots, n).$$

From $\theta_{x_i p_k}$,

(7) $$\mathbf{S}(X_k + p_k Z) F_{X_i} - \mathbf{S} P_i F_{P_k} = 0 \qquad (i=1, \cdots, n;\ k=1, \cdots, n).$$

From θ_{zz},

(8) $$\mathbf{S}\left(\sum_{\rho=1}^{n} p_\rho P_\rho\right) F_Z = 0.$$

From $\theta_{z p_i}$,

(9) $$\mathbf{S}(X_i + p_i Z) F_Z - \mathbf{S}\left(\sum_{\rho=1}^{n} p_\rho P_\rho\right) F_{P_i} = 0 \qquad (i=1, \cdots, n).$$

From $\theta_{p_i p_k}$,

(10) $$\mathbf{S}(X_k + p_k Z) F_{P_i} + \mathbf{S}(X_i + p_i Z) F_{P_k} = 0. \qquad (i, k=1, \cdots, n).$$

It follows from these equations that F must be a function of the variables

$$A_{1,1}, \cdots, A_{1,n}, A_{2,1}, \cdots, A_{2,n}, \cdots, Z_1, Z_2, \cdots, Z_r, P_{1,1}, \cdots, P_{r,n},$$

where

$$A_{i,k} \equiv X_{i,k} + p_k Z_i \qquad (i=1, \cdots, r;\ k=1, \cdots, n).$$

If we modify the system of equations by assuming F to be a function of these variables only, it becomes

(11) $$\mu(n+1)F = \mathbf{S} Z F_Z + \mathbf{S} \sum_{i=1}^{n} P_i F_{P_i},$$

(12) $$\mathbf{S} P_i F_{A_k} + \mathbf{S} P_k F_{A_i} = 0 \qquad (i, k=1, \cdots, n),$$

(13) $$\mathbf{S} P_i F_Z = 0 \qquad (i=1, \cdots, n),$$

(14) $$\mathbf{S} A_k F_{A_i} - \mathbf{S} P_i F_{P_k} = 0 \qquad (i, k=1, \cdots, n),$$

(15) $$\mathbf{S} A_i F_Z = 0 \qquad (i=1, \cdots, n),$$

(16) $$\mathbf{S} A_i F_{P_k} + \mathbf{S} A_k F_{P_i} = 0 \qquad (i, k=1, \cdots, n).$$

If we put aside for the present the first of these equations, the remaining equations form a complete system. The number of functionally independent solu-

tions is, however, not immediately deducible, as some of the equations may depend algebraically on the others. In particular, equations (13) and (15) show that if the number r of expressions f considered is not greater than $2n$, and if all the determinants of order r of the matrix

$$\begin{matrix} A_{1,1} & A_{1,2} & \cdots & A_{1,n} & P_{1,1} & \cdots & P_{1,n} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,n} & P_{2,1} & \cdots & P_{2,n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ A_{r,1} & A_{r,2} & \cdots & A_{r,n} & P_{r,1} & \cdots & P_{r,n} \end{matrix}$$

do not vanish, F_Z must be zero.

Further, there is no necessity to consider more than $2n+1$ expressions f, for if there were more than this number they could all be expressed by means of any $(2n+1)$ of them.

Suppose that there is only one expression f, then F_{A_i}, $F_{\dot{Z}}$, F_{P_i} $(i=1, \cdots, n)$ are all zero, and therefore it follows that there is no invariant of the type sought of a single expression f.

Next let there be two expressions f. Then F_{Z_1}, F_{Z_2} are zero unless

$$\frac{A_{1,1}}{A_{2,1}} = \frac{A_{1,2}}{A_{2,2}} = \cdots = \frac{P_{1,i}}{P_{2,i}} = \cdots.$$

But these relations are equivalent to the conditions that any equation $\phi(x_i z p_i)=0$ which lies in involution with $f_1=a$, shall lie in involution with $f_2=b$, where a and b are arbitrary constants.

Now corresponding to a particular value of a there are ∞^{2n-1} characteristic strips which go to build up the integrals of the equation $f_1=a$.

Hence taking account of all values of a, all the surface elements (zx_ip_i) in space of $(n+1)$ dimensions are arranged in ∞^{2n} characteristic strips.

The condition given above is easily seen to be the same as the condition that $f_2=b$ determines the same system of ∞^{2n} characteristic strips.

Exactly similarly in the case of r expressions f, there are obtained ∞^{2n} $(2n-r+1)$-fold manifolds which are common to the first $(r-1)$ equations $f_1=a_1, \cdots, f_{r-1}=a_{r-1}$, where $a_1, \cdots, a_{r-1}$ are arbitrary constants, and the conditions in virtue of which $F_{Z_1}, F_{Z_2}, \cdots, F_{Z_r}$ are not zero are the conditions that these manifolds should also satisfy $f_r=a_r$ for all values of the constant a_r.

We accordingly neglect this particular case, and then all the differential coefficients F_Z are zero, provided $r<2n+1$.

The system of equations which F satisfies is readily integrated, and it is seen that F must be a function of the Poisson alternants

$$[f_\lambda f_\mu] \qquad (\lambda, \mu = 1, 2 \cdots r),$$

and in addition must satisfy equation (11). This equation merely expresses the

fact that F must be homogeneous of degree $\mu(n+1)$ in the quantities P, and F is therefore a homogeneous function of the quantities

$$[f_\lambda f_\mu] \qquad (\lambda, \mu = 1, 2 \cdots r).$$

There remains one case still to be considered, namely that in which we have $(2n+1)$ expressions f. In this case there exist invariants of the form $[f_\lambda f_\mu](\lambda, \mu = 1, 2, \cdots, 2n-1)$ and among these the only relations are those of the type $[f_\lambda f_\mu] + [f_\mu f_\lambda] = 0$.

We therefore have $n(2n+1)$ functionally independent solutions of our system of equations.

But returning to this system, we find that it consists of $2n^2 + 3n$ equations in addition to an equation which expresses a condition of homogeneity. There are $(2n+1)^2$ variables involved in the equations, and the equations are now algebraically independent. They therefore possess $(2n+1)^2 - (2n^2+3n)$ functionally independent solutions. Of this number, $(2n^2+n+1)$, $n(2n+1)$ are accounted for, and therefore one solution still remains to be discovered.

It is readily seen that this solution is

$$\begin{vmatrix} Z_1 & A_{1,1} & A_{1,2}, \cdots & P_{1,1} & P_{1,2}, \cdots & P_{1,n} \\ Z_2 & A_{2,1} & \cdots & & & P_{2,n} \\ \vdots & & & & & \\ Z_{2n+1} & A_{2n+1,1} & & & & P_{2n-1,n} \end{vmatrix} \equiv J.$$

If we substitute in (11) we see that, if F is J, μ is equal to unity.

Collecting results we see that *the only functionally independent relative invariants of our type, of r expressions f, are the alternants $[f_\lambda f_\mu]$ if r is less than $2n+1$, and if r is equal to $(2n+1)$ there is one additional invariant, the Jacobian of the forms with respect to the variables involved in them.*

It is well known that the alternants $[f_\lambda f_\mu]$ are all invariants of the forms f. The theorem that these are the only invariants of the type sought, has been given by Lie,* who, however, merely suggests the method of proof. Further, Lie has apparently overlooked the additional invariant which arises in connection with $(2n+1)$ forms though he must have been perfectly familiar with the fact that the Jacobian is an invariant.

§ 2.

Let us now consider expressions involving one dependent variable, n independent variables, and the derivatives of the dependent variable with respect to the independent ones of the first and second orders.

The variables involved are now

$$z, x_1, x_2, \cdots, x_n, p_1, p_2, \cdots, p_n, \cdots, p_{ik} \qquad (i, k = 1, 2 \cdots n).$$

* Lie, Mathematische Annalen, vol. 24 (1884), p. 578, and Göttingen Nachrichten (1872), pp. 478–479.

The invariant sought will be a function of these, and of the first derivatives of the expression with respect to the variables involved in them.

Assume that the infinitesimal contact transformation is determined as before by a function θ, and that F is the invariant. Then the equation satisfied by F is

$$\frac{dF}{dt} + \left\{ \mu_0 (n+1) \frac{\partial \theta}{\partial z} + \mu_1 \left[\frac{n(n+1)}{2} \frac{\partial \theta}{\partial z} + (n+1) \sum_{h=1}^{m} \frac{d}{dx_h} \left(\frac{\partial \theta}{\partial p_h} \right) \right] \right\} F = 0 .$$

Expand this, and it becomes

$$F_z \frac{dz}{dt} + \sum_{r=1}^{n} F_{x_r} \frac{dx_r}{dt} + \sum_{r=1}^{n} F_{p_r} \frac{dp_r}{dt} + \sum_{\alpha\beta} F_{p_{\alpha\beta}} \frac{dp_{\alpha\beta}}{dt} + \mathbf{S} F_Z \frac{dZ}{dt} + \mathbf{S} \sum_{r=1}^{n} F_{X_r} \frac{dX_r}{dt}$$
$$+ \mathbf{S} \sum_{r=1}^{n} F_{P_r} \frac{dP_r}{dt} + \mathbf{S} \sum_{\alpha\beta} F_{P_{\alpha\beta}} \frac{dP_{\alpha\beta}}{dt} + \left[\lambda \theta_z + \mu \sum_{r=1}^{n} \frac{d}{dx_r} (\theta_{p_r}) \right] F = 0,$$

where

$$\lambda = (n+1) \left(\mu_0 + \frac{n}{2} \mu_1 \right),$$

$$\mu = (n+1) \mu_1 .$$

As before, F is an invariant to all contact transformations, and therefore, if we substitute in the above equation the values of

$$\frac{dz}{dt}, \frac{dx_r}{dt}, \cdots, \frac{dZ}{dt}, \cdots$$

in terms of θ and its derivatives, and if we then equate to zero the coefficients of the various derivatives of θ, we obtain a system of linear differential equations which F must satisfy.

If we equate to zero the coefficients of

$$\theta,\ \theta_{x_1},\ \cdots,\ \theta_{x_n},\ \theta_{x_1 x_1},\ \cdots,\ \theta_{x_n x_n},\ \theta_{p_1},\ \cdots,\ \theta_{p_n},$$

we obtain the following system of equations:

$$F_z = 0, \qquad F_{x_i} = 0 \qquad (i = 1, 2, \cdots, n),$$
$$F_{p_i} - \mathbf{S} Z F_{X_i} = 0 \qquad (i = 1, 2, \cdots, n),$$
$$F_{p_{ii}} - \mathbf{S} P_i F_{X_i} = 0 \qquad (i = 1, 2, \cdots, n),$$
$$F_{p_{ik}} - \mathbf{S} P_i F_{X_k} - \mathbf{S} P_k F_{X_i} = 0 \qquad (i, k = 1, 2, \cdots, n, i \neq k).$$

We therefore introduce the variables

$$A_i = X_i + p_i Z + \sum_{k=1}^{n} p_{ik} P_k \qquad (i = 1, 2, \cdots, n),$$

and then these equations show that F is a function of

$$A_1, \cdots, A_n, Z, P_1, \cdots, P_n, P_{11}, P_{12}, P_{nn},$$

only.

The increments of the A's due to the transformation are readily found, and we have

$$-\frac{dA_i}{dt} = -\sum_{k=1}^{n} A_k \left(\frac{d\theta_{p_k}}{dx_i}\right) + \sum_{jk} \left\{\frac{d}{dx_i}\left(\frac{d^2\theta}{dx_j\,dx_k}\right)\right\} P_{jk},$$

where $(d^2\theta/dx_\alpha dx_\beta)$ denotes as before the differential coefficient of θ with respect to x_α and x_β, in which $z, p_r, p_{\alpha\beta}$ are taken to be functions of the x's, the terms containing third derivatives of z being omitted, and d/dx_i is written for

$$\frac{\partial}{\partial x_i} + p_i \frac{\partial}{\partial z} + \sum_{k=1}^{n} p_{ik} \frac{\partial}{\partial p_k}.$$

It is easy to show that

$$\frac{d}{dx_i}\left(\frac{d^2\theta}{dx_\alpha\,dx_\beta}\right) = \frac{d}{dx_\alpha}\left(\frac{d^2\theta}{dx_i\,dx_\beta}\right) = \frac{d}{dx_\beta}\left(\frac{d^2\theta}{dx_i\,dx_\alpha}\right) = \left(\left(\frac{d^3\theta}{dx_i\,dx_\alpha\,dx_\beta}\right)\right),$$

where the last expression denotes $d^3\theta/dx_i dx_\alpha dx_\beta$ with the terms containing third and fourth derivatives of z omitted.

The increments expressed in the variables A, Z, P are therefore as follows:

$$-\frac{dA_i}{dt} = -\sum_{k=1}^{n} A_k \frac{d\theta_{p_k}}{dx_i} + \sum_{jk} \left(\left(\frac{d^3\theta}{dx_i\,dx_j\,dx_k}\right)\right) P_{jk},$$

$$-\frac{dZ}{dt} = -\sum_{k=1}^{n} A_k \frac{\partial^2\theta}{\partial p_k \partial z} + Z\theta_z + \sum_{k=1}^{n} P_k \frac{d\theta_z}{dx_k} + \sum_{jk} \frac{\partial}{\partial z}\left(\frac{d^2\theta}{dx_j\,dx_k}\right) P_{jk},$$

$$-\frac{dP_i}{dt} = -\sum_{k=1}^{n} A_k \frac{\partial^2\theta}{\partial p_i \partial p_k} + \sum_{k=1}^{n} P_k \frac{d\theta_{p_i}}{dx_k} + P_i\theta_z + \sum_{jk} \frac{\partial}{\partial p_i}\left(\frac{d^2\theta}{dx_j\,dx_k}\right) P_{jk},$$

$$-\frac{dP_{hk}}{dt} = \sum_{\alpha\beta} \frac{\partial}{\partial p_{hk}}\left(\frac{d^2\theta}{dx_\alpha dx_\beta}\right) P_{\alpha\beta}.$$

If F is taken as a function of $A_1, \cdots, A_n, Z, P_1, \cdots, P_n, P_{11}, P_{12}, \cdots, P_{nn}$, the equation

$$\frac{dF}{dt} + \left[\lambda\theta_z + \mu \sum_k \frac{d\theta_{p_k}}{dx_k}\right] F = 0$$

becomes

$$\begin{aligned}
&\mathbf{S} \sum_{i=1}^{n} F_{A_i} \left\{ \sum_{k=1}^{n} A_k \frac{d\theta_{p_k}}{dx_i} - \sum_{jk} P_{jk} \left(\left(\frac{d^3\theta}{dx_i\,dx_j\,dx_k}\right)\right)\right\} \\
&+ \mathbf{S} F_Z \left\{ \sum_{k=1}^{n} A_k \theta_{p_k z} - Z\theta_z - \sum_{k=1}^{n} \frac{d\theta_z}{dx_k} P_k - \sum_{jk} P_{jk} \frac{\partial}{\partial z}\left(\frac{d^2\theta}{dx_j\,dx_k}\right)\right\} \\
&+ \mathbf{S} \sum_{i=1}^{m} F_{P_i} \left\{ \sum_{r=1}^{n} A_r \theta_{p_i p_r} - \sum_{r=1}^{n} P_r \frac{d\theta_{p_i}}{dx_r} - P_i\theta_z + \sum_{jk} P_{jk} \frac{\partial}{\partial p_i}\left(\frac{d^2\theta}{dx_j\,dx_k}\right)\right\} \\
&- \mathbf{S} \sum F_{P_{hk}} \left\{ \sum_{\alpha\beta} P_{\alpha\beta} \frac{\partial}{\partial p_{hk}}\left(\frac{d^2\theta}{dx_\alpha dx_\beta}\right)\right\} + \left[\lambda\theta_z + \mu \sum_{k=1}^{n} \frac{d\theta_{p_k}}{dx_k}\right] F = 0.
\end{aligned}$$

Equating coefficients of the derivatives of θ to zero, we obtain the following system of equations:

$$\mathbf{S}\left[P_{jk}F_{A_i}+P_{ki}F_{A_j}+P_{ij}F_{A_k}\right]=0 \quad (i\neq j,\ j\neq k,\ k\neq i) \tag{17}$$

$$\mathbf{S}\left[P_{ij}F_{A_i}+P_{ii}F_{A_j}\right]=0, \tag{18}$$

$$\mathbf{S}\left[P_{ij}F_{Z}\right]=0, \tag{19}$$

$$\mathbf{S}\left[P_{ij}F_{P_k}\right]=0, \tag{20}$$

$$\mathbf{S}\left[A_iF_{A_i}-P_iF_{P_i}-P_{ii}F_{P_{ii}}-\sum_{j=1}^{n}P_{ij}F_{P_{ij}}\right]+\mu F=0, \tag{21}$$

$$\mathbf{S}\left[A_iF_{A_j}-P_jF_{P_i}-\sum_{k=1}^{n}P_{kj}F_{P_{ki}}-P_{jj}F_{P_{ji}}\right]=0 \quad (i\neq j), \tag{22}$$

$$\mathbf{S}\left[P_iF_Z\right]=0, \tag{23}$$

$$\mathbf{S}\left[A_iF_Z\right]=0, \tag{24}$$

$$\mathbf{S}\left[A_jF_{P_i}+A_iF_{P_j}\right]=0, \tag{25}$$

$$\mathbf{S}\left[ZF_Z+\sum_{i=1}^{n}P_iF_{P_i}+\sum_{\alpha\beta}P_{\alpha\beta}F_{P_{\alpha\beta}}\right]=\lambda F \quad (i,j,k=1,2,\cdots,n). \tag{26}$$

There now arise two cases to be considered. In the first case $P_{ij}=0$ $(i,j=1,2,\cdots,n)$, and our expressions are therefore of the first order. In this case the equations become

$$\mathbf{S}\left[A_iF_{A_i}-P_iF_{P_i}\right]+\mu F=0,$$

$$\mathbf{S}\left[A_iF_{A_j}-P_jF_{P_i}\right]=0,$$

$$\mathbf{S}\left[P_iF_Z\right]=0,\ \mathbf{S}\left[A_iF_Z\right]=0,$$

$$\mathbf{S}\left[A_jF_{P_i}+A_iF_{P_j}\right]=0,$$

$$\mathbf{S}\left[ZF_Z+\sum_{i=1}^{n}P_iF_{P_i}\right]=\lambda F \qquad (i,j=1,2,\ldots n).$$

These equations are almost identical with the set (11)...(16); they must of course possess the same integrals as that set, together with others arising from the facts that equations of the type (12) have not now to be satisfied, and that μ, which is an arbitrary constant, has the particular value zero in the first set of equations.

It is easy to see that the integrals still to be found are functions of the variables A alone, and therefore they satisfy the equations

$$\mathbf{S}A_iF_{A_i}+\mu F=0,$$

$$\mathbf{S}A_iF_{A_i}=0 \qquad (i,j\ -1,2,\cdots,n).$$

Assuming that there are r expressions of the first order whose invariants we are seeking, the solutions of the above equations are easily seen to be the n-row determinants of the matrix,

$$\begin{Vmatrix} A_{1,1} & A_{2,1} & \cdots & A_{r,1} \\ A_{1,2} & & \cdots & \vdots \\ \vdots & & & \\ A_{1,n} & & \cdots & A_{r,n} \end{Vmatrix},$$

provided that r is greater than n.

There are no additional solutions if r is less than n.

These solutions are the Jacobians of sets of n of the forms.

We shall now consider the case in which the quantities P_{ij} are not all zero.

Before discussing the general case, we shall consider the case in which there are only two independent variables.

We shall take in order the cases in which there are one, two, three expressions of the second order whose invariants we are seeking.

In the case of one such expression,

$$F_{A_1} = F_{A_2} = 0 = F_Z = F_{P_1} = F_{P_2},$$

$$2P_{11} F_{P_{11}} + P_{12} F_{P_{12}} = \mu F,$$

$$P_{12} F_{P_{12}} + 2P_{22} F_{P_{22}} = \mu F,$$

$$2P_{11} F_{P_{12}} + P_{21} F_{P_{22}} = 0,$$

$$2P_{22} F_{P_{12}} + P_{21} F_{P_{11}} = 0.$$

These equations show that F must be a homogeneous function of the algebraic invariant of the binary form

$$(P_{11},\ \tfrac{1}{2}P_{12},\ P_{22} \between * \ )^2.$$

Hence $F \equiv \text{const.} \times (P_{12}^2 - 4P_{11}P_{22})^{\mu/2}$.

To interpret this invariant, suppose that

$$f(x_1\ x_2\ z\ p_1\ p_2\ p_{11}\ p_{12}\ p_{22}) = 0$$

is a differential equation of the second order.

Let $z = \phi(x_1 x_2)$ be some non-singular solution of this equation. We define two directions on this integral surface by means of the equation

$$P_{22} dx_1^2 - P_{12} dx_1 dx_2 + P_{11} dx_2^2 = 0.$$

Along one of the curves thus determined on the particular integral surface, x_1, x_2, z, p_1, p_2, p_{11}, p_{12}, p_{22} are functions of a single parameter while dz, dp_1, dp_2 are determined from the equations

$$dz = p_1 dx_1 + p_2 dx_2,$$

$$dp_1 = p_{11} dx_1 + p_{12} dx_2,$$

$$dp_2 = p_{12} dx_1 + p_{22} dx_2.$$

Also

$$\frac{df}{dx_1} + P_{11}\frac{dp_{11}}{dx_1} + P_{22}\frac{dp_{12}}{dx_2} = 0,$$

$$\frac{df}{dx_2} + P_{11}\frac{dp_{12}}{dx_1} + P_{22}\frac{dp_{22}}{dx_2} = 0.$$

The system of seven equations thus obtained are equivalent to six distinct relations, and they determine the "characteristics" of the given equation $f = 0$.

The fundamental property in connection with the curves obtained is that if two integral surfaces have contact of the first order along a characteristic, and if they have contact of the second order at any one point of this curve, they have contact of the second order all along the curve.*

We notice that the directions of the curves are given at every point by means of $z = \phi(x_1 x_2)$ and $P_{11} dx_2^2 - P_{12} dx_1 dx_2 + P_{22} dx_1^2 = 0$.

Now the transformation considered changes an integral surface into an integral surface, and also a characteristic upon an integral surface into a characteristic upon the transformed surface. We therefore expect $P_{11} dx_2^2 - \cdots = 0$ to be an invariant of the expression considered, and we also expect any function geometrically connected with it to be an invariant. $P_{12}^2 - 4P_{11}P_{22}$ was therefore *a priori* to be expected as an invariant.

We next consider the case of two expressions f_1 and f_2.

It is easy to see that the only invariants are the algebraic invariants of the two binary forms

$$(P_{1,11}\ \tfrac{1}{2}P_{1,12}\ P_{1,22} \between\ *\ )^2,$$

$$(P_{2,11}\ \tfrac{1}{2}P_{2,12}\ P_{2,22} \between\ *\ )^2,$$

unless the condition

$$I \equiv \begin{vmatrix} P_{1,12} & P_{2,12} \\ P_{1,11} & P_{2,11} \end{vmatrix} \times \begin{vmatrix} P_{1,22} & P_{2,22} \\ P_{1,12} & P_{2,12} \end{vmatrix} - \begin{vmatrix} P_{1,11} & P_{2,11} \\ P_{1,22} & P_{2,22} \end{vmatrix}^2 = 0$$

holds.

These invariants have an immediate interpretation from the theory of characteristics. I is itself one of the invariants above mentioned, and $I = 0$ is the condition that the two quadratic forms above mentioned, when equated to zero, have a common root. Therefore, unless the characteristics of $f_1 = 0$ have one direction common with those of $f_2 = 0$ at every point on a common integral surface, the two expressions have only three functionally independent invariants of

* GOURSAT, *Equations aux dérivées partielles du second ordre*, vol. 1, pp. 170 seq.

our type, namely those of two quadratic forms. Suppose now that I is zero. Let m be used to denote the common root mentioned above. Then

$$m = -\frac{P_{1,11}P_{2,22} - P_{2,11}P_{1,22}}{P_{1,12}P_{2,11} - P_{2,12}P_{1,11}} = -\frac{P_{1,22}P_{2,12} - P_{2,22}P_{1,12}}{P_{1,11}P_{2,22} - P_{2,11}P_{1,22}}.$$

Let H denote $A_{1,1}P_{2,11} - A_{2,1}P_{1,11}$ and let K denote $A_{1,2}P_{2,22} - A_{2,2}P_{1,22}$. Then it easily follows that $K + mH$ satisfies our system of equations provided it is zero, and further, this is the only additional solution the system can have.

We may verify that

$$\frac{d}{dt}(K + mH) = \left\{ 3\theta_z - 3\frac{d\theta_{p_1}}{dx_1} + m\frac{d\theta_{p_1}}{dx_2} + \frac{1}{m}\frac{d\theta_{p_2}}{dx_1} - \frac{d\theta_{p_2}}{dx_1} \right\} \times (K + mH).$$

Hence $K + mH$ is not an invariant of our type, although $K + mH = 0$ is an invariant relation.

These two equations

$$I = 0,$$

$$K + mH = 0$$

have an important signification in the theory of differential equations. They are* the conditions that the two equations $f_1 = 0$, $f_2 = 0$, form a system in involution, in other words, they are the conditions that the two equations have a system of common integrals depending on an infinite number of arbitrary constants.

We shall now consider invariants of three expressions f_1, f_2, f_3.

From the system of equations it follows that $F_Z = 0$ unless all the 3-row determinants of the matrix

$$\begin{array}{ccccccc} A_{1,1} & A_{1,2} & P_{1,1} & P_{1,2} & P_{1,11} & P_{1,12} & P_{1,22} \\ A_{2,1} & A_{2,2} & P_{2,1} & P_{2,2} & P_{2,11} & P_{2,12} & P_{2,22} \\ A_{3,1} & A_{3,2} & P_{3,1} & P_{3,2} & P_{3,11} & P_{3,12} & P_{3,22} \end{array}$$

are zero.

Also F_{P_λ} is zero unless all the 3-row determinants of the matrix

$$\left|\begin{array}{ccccc} A_{1,1} & A_{1,2} & P_{1,11} & P_{1,12} & P_{1,22} \\ A_{2,1} & A_{2,2} & P_{2,11} & P_{2,12} & P_{2,22} \\ A_{3,1} & A_{3,2} & P_{3,11} & P_{3,12} & P_{3,22} \end{array}\right|$$

are zero.

Assuming that the conditions mentioned are not satisfied we see that

$$F_Z = 0, \qquad F_{P_1} = 0, \qquad F_{P_2} = 0.$$

---

* GOURSAT, *Equations aux dérivées partielles du second ordre*, vol. 2, p. 40 and p. 76.

The functionally independent solutions of our equations are then seen to be the algebraic invariants of the binary forms $(K, H \between *)$:

$$(P_{1,11},\ \tfrac{1}{2}P_{1,12},\ P_{1,22} \between *)^2,$$
$$(P_{2,11},\ \tfrac{1}{2}P_{2,12},\ P_{2,22} \between *)^2,$$
$$(P_{3,11},\ \tfrac{1}{2}P_{3,12},\ P_{3,22} \between *)^2,$$

where

$$H \equiv \begin{vmatrix} P_{1,11} & P_{2,11} & P_{3,11} \\ A_{1,1} & A_{2,1} & A_{3,1} \\ P_{1,22} & P_{2,22} & P_{3,22} \end{vmatrix} - \begin{vmatrix} A_{1,2} & \cdots \\ P_{1,12} & \cdots \\ P_{1,22} & \cdots \end{vmatrix},$$

$$K \equiv \begin{vmatrix} P_{1,11} & \cdots \\ A_{1,2} & \cdots \\ P_{1,22} & \cdots \end{vmatrix} - \begin{vmatrix} P_{1,11} & \cdots \\ P_{1,12} & \cdots \\ A_{1,1} & \cdots \end{vmatrix}.$$

It is important to notice the meaning of the equations $H = 0$, $K = 0$. They are in fact the conditions that the three equations $f_1 = 0$, $f_2 = 0$, $f_3 = 0$ have a common integral surface.*

The three quadratic binary forms and their invariants have as before, immediate interpretation from the theory of characteristics, but the linear form $K dx_2 - H dx_1$ has no such immediate interpretation.

As another example of invariants of this type we shall now consider the case in which there are two expressions, one of the second order and the other of the first.

The equations are readily solved, and the solutions are the algebraic invariants of the two forms

$$(P_{1,11}, \tfrac{1}{2} P_{1,12}, P_{1,22} \between *)^2,$$
$$(A_{2,2}, - A_{2,1} \between *),$$

where $P_{2,11}$, $P_{2,12}$, $P_{2,22}$ are all zero. We know that

$$P_{1,11} dx_2^2 - P_{1,12} dx_1 dx_2 + P_{1,22} dx_1^2 = 0$$

is the equation for the directions of the characteristics of $f_1 = 0$. It seems, therefore, important to consider the meaning of

$$A_{2,2} dx_2 - (- A_{2,1}) dx_1,$$

or

$$A_{2,1} dx_1 + A_{2,2} dx_2.$$

But this is equal to df_2, provided

$$dz - p_1 dx_1 - p_2 dx_2 = 0, \quad dp_1 - p_{11} dx_1 - p_{12} dx_2 = 0, \quad dp_2 - p_{12} dx_1 - p_{22} dx_2 = 0.$$

* VALYI, Crelle's Journal, vol. 95 (1883) p. 100; GOURSAT, loc. cit., vol. 2, p. 199.

We thus have an interpretation of both invariants.

We now return to the general case, when there are n independent variables. The equations (17)—(26) possess all the invariants of our type as solutions.

Suppose that there are r expressions f. Then from (19), (23), (24), $F_Z = 0$ unless r is greater than $\frac{1}{2}n(n+1)+2n$, or unless all the r-row determinants of the matrix

$$\left\| \begin{matrix} P_{1,11} & P_{1,12} & \cdots & P_{1,nn} & P_{1,1} & \cdots & P_{1,n} & A_{1,1} & \cdots & A_{1,n} \\ \vdots & \vdots & & & & & & & & \\ P_{r,11} & P_{r,12} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & A_{r,n} \end{matrix} \right\|$$

vanish.

Also the equations (17) (18) show that unless the determinants of another matrix vanish, $F_{A_i} = 0$ for all values of i.

Suppose that $F_Z, F_{A_1}, \ldots, F_{A_n}$ are all zero. There remains the system of equations for F

$$\mathbf{S}(P_{ij} F_{P_i}) = 0,$$

$$\mathbf{S}\left[P_i F_{P_i} + P_{ii} F_{P_{ii}} + \sum_{i=1}^{n} P_{ij} F_{P_{ij}}\right] - \mu F = 0,$$

$$\mathbf{S}\left[P_j F_{P_i} + \sum_{k=1}^{n} P_{kj} F_{P_{ki}} + P_{jj} F_{P_{ji}}\right] = 0,$$

$$\mathbf{S}\left[A_j F_{P_i} + A_i F_{P_j}\right] = 0,$$

$$\mathbf{S}\left[\sum_{i=1}^{n} P_i F_{P_i} + \sum_{\alpha\beta} P_{\alpha\beta} F_{P_{\alpha\beta}}\right] = \lambda F \qquad (i, j, k = 1, \cdots, n),$$

and the fourth of these equations shows that F_{P_i} is zero for all values of i. Hence the system becomes

$$\mathbf{S}\left[P_{ii} F_{P_{ii}} + \sum_{j=1}^{n} P_{ij} F_{P_{ij}}\right] - \mu F = 0,$$

$$\mathbf{S}\left[\sum_{k=1}^{n} P_{kj} F_{P_{ki}} + P_{jj} F_{P_{ji}}\right] = 0,$$

$$\mathbf{S}\left[\sum_{\alpha\beta} P_{\alpha\beta} F_{P_{\alpha\beta}}\right] = \lambda F \qquad (i, j, k = 1, \cdots, n).$$

Hence $\lambda = n\mu/2$, and the invariants required are the algebraic invariants of the system of quadratic forms

$$\sum_{\alpha\beta} P_{\alpha\beta} x_\alpha x_\beta,$$

where none of the magnitudes $P_{\alpha\beta}$ is repeated and $P_{\alpha\beta} = P_{\beta\alpha}$.

These quadratic forms are easily seen to be those which FORSYTH * calls

* FORSYTH, Philosophical Transactions, ser. A, vol. 191 (1898), p. 2.

"characteristic invariants" when there are only three independent variables. This name might with advantage be extended to the general case in which there are n independent variables.

We now return to the case in which $F_{A_1} F_{A_2}$, etc., are not all zero.

Suppose that there are three independent variables, and two expressions f. Then if F_{A_1}, etc., are not zero, all the 6-row determinants of the matrix

$$\left\| \begin{matrix} P_{1,11} & 0 & 0 & P_{1,12} & P_{1,22} & 0 & 0 & P_{1,31} & P_{1,33} & P_{1,23} \\ P_{2,11} & 0 & 0 & P_{2,12} & P_{2,22} & 0 & 0 & P_{2,31} & P_{2,33} & P_{2,23} \\ 0 & P_{1,22} & 0 & P_{1,11} & P_{1,12} & P_{1,23} & P_{1,33} & 0 & 0 & P_{1,31} \\ 0 & P_{2,22} & 0 & P_{2,11} & P_{2,12} & P_{2,23} & P_{2,33} & 0 & 0 & P_{2,31} \\ 0 & 0 & P_{1,33} & 0 & 0 & P_{1,22} & P_{1,23} & P_{1,11} & P_{1,31} & P_{1,12} \\ 0 & 0 & P_{2,33} & 0 & 0 & P_{2,22} & P_{2,23} & P_{2,11} & P_{2,31} & P_{2,12} \end{matrix} \right\|$$

must vanish.

Let S_i denote the characteristic invariant of f_i, then it is easily seen that if we construct the cubic forms

$$S_1 x_1, \; S_1 x_2, \; S_1 x_3, \; S_2 x_1, \; S_2 x_2, \; S_2 x_3,$$

the above matrix is the matrix of the coefficients.

Hence, if our conditions hold, the above six cubics must belong to a five fold linear system. Expressing this condition we see that $S_1 L_1 + S_2 L_2 \equiv 0$, where L_1 and L_2 are certain linear forms.

Hence either S_1 and S_2 both break up into linear factors, or S_2 is equivalent to S_1.

In the case in which there are three expressions f, the conditions give

$$S_1 L_1 + S_2 L_2 + S_3 L_3 \equiv 0,$$

where L_1, L_2, L_3 are linear. Hence, in general S_1, S_2, S_3, regarded as conics, have two common points.

The generalization is immediate, and the condition in order that invariants involving the magnitudes A, of r expressions f in n independent variables exist, are equivalent to the conditions that r linear forms $L_1, \cdots, L_r$ should exist such that

$$\sum_{i=1}^{r} S_i L_i \equiv 0$$

identically, when S_i is the characteristic invariant of f_i. It is readily seen that these conditions may be expressed by the vanishing of certain algebraic invariants of the r quadratic forms S.

An upper limit to the number of these conditions may readily be obtained.

This upper limit is $n(n+1)(n+2)/3! - nr + 1$. The number may fall below this in certain cases, for example, if $n = 3$ and $r = 2$ it is 4, whilst

$$\frac{n(n+1)(n+2)}{3!} - nr + 1 = 5.$$

If all the linear expressions L are equivalent, the conditions require that constants λ can be found such that $\sum \lambda S = 0$.

The number of conditions for this is readily seen to be

$$\frac{n(n+1)}{2} - r + 1.$$

This number is less than the previous one if $[n(n+1)/6 - r](n-1) > 0$, and the second conditions are all independent.

Hence, if r is $< \frac{1}{6}n(n+1)$ the imposition of $n(n+1)/2 - r + 1$ conditions is sufficient to cause the remainder to be satisfied.

Further consideration of this question will be omitted from the present paper.

Suppose the conditions in question to be satisfied. We then obtain a solution of the set of equations in F_A which is a determinant linear in the magnitudes A. Call this determinant Δ, then in a manner strictly analogous to the case when $n = 2$, it may be shown that $\Delta = 0$ is an invariant relation, provided the previous set of conditions holds.

If there are solutions of the system of equations considered which involve the magnitudes A, the derivatives F_{P_i} are not necessarily zero. From the equations of type (20), we see that, if $F_{P_i} \neq 0$, the matrix

$$\left\| \begin{matrix} P_{1,11} & \cdots & P_{1,ij} & \cdots & P_{1,nn} \\ \vdots & & \vdots & & \\ P_{r,11} & \cdots & P_{r,ij} & \cdots & P_{r,nn} \end{matrix} \right\|$$

must have all its r-row determinants zero.

In addition, all the equations of type (25),

$$\mathbf{S}(A_j F_{P_i} + A_i F_{P_j}) = 0,$$

must be satisfied.

Now take any one of the determinants of the above matrix, replace one of its columns by the magnitudes $P_{1,i}, P_{2,i}, \cdots, P_{r,i}$. Call the determinant then formed Δ_i. Call the similar determinant with A_i instead of P_i, M_i. Then P_i only enters through Δ_i, and the equations (25) become

$$M_j F_{\Delta_i} + M_i F_{\Delta_j} = 0 \qquad (i, j = 1, \cdots, r).$$

Hence if an invariant contains P_i, we see from the case when $i = j$, that M_i must be zero. If $M_i = 0$ and $F_{\Delta_i} \neq 0$, we see that $M_j = 0$ $(j = 1, \cdots, r)$; and if $F_{\Delta_i} = 0$ and $M_i \neq 0$, we see that $F_{\Delta_j} = 0$ $(j = 1, 2, \cdots, r)$.

Hence either none of the magnitudes $P_i (i = 1, \cdots, r)$ occur in any invariant, or all the magnitudes $M_i (i = 1, \cdots, r)$ are zero.

It is easy to see that if all the given conditions are satisfied, then the magnitudes Δ_i satisfy the remaining equations, and therefore these magnitudes Δ_i are invariants.

There is no invariant involving Z unless all the conditions given in connection with the magnitude P_i are satisfied and, in addition,

$$\Delta_1 = \Delta_2 = \cdots = \Delta_r = 0.$$

If all these conditions are satisfied, there is an invariant involving Z given by replacing any column in any r-row determinant of the matrix

$$\left\| \begin{matrix} P_{1,11} & \cdots & P_{1,ij} & \cdots & P_{1,nn} \\ \vdots & & \vdots & & \vdots \\ P_{r,11} & \cdots & P_{r,ij} & \cdots & P_{r,nn} \end{matrix} \right\|$$

by $Z_1, Z_2, \cdots, Z_r$.

§ 3.

We have not as yet considered invariants which involve the magnitudes $dx_i, dz, dp_i, dp_{ij} (i, j = 1, \cdots, n)$.

It is clear that invariants of this type do exist. For example, it is easy to verify that

$$\sum_{\alpha, \beta} P_{\alpha\beta}\, dx_\alpha\, dx_\beta$$

is such an invariant.

The work is somewhat simplified if we take as variables

$$\left.\begin{aligned} dx_i &\equiv a_i, \\ dz - \sum_i p_i\, dx_i &\equiv u, \\ dp_i - \sum_j p_{ij}\, dx_j &\equiv v_i, \\ dp_{ij} &\equiv e_{ij}. \end{aligned}\right\} \qquad (i, j = 1, \cdots, n).$$

The increments of these magnitudes are readily obtained. We have

$$-\frac{da_i}{dt} = \sum_k \frac{d\theta_{p_i}}{dx_k} a_k + \frac{\partial \theta_{p_i}}{\partial z} u + \sum_k \frac{\partial^2 \theta}{\partial p_i \partial p_k} v_k, \qquad \frac{du}{dt} = \theta_z u,$$

$$\frac{dv_i}{dt} = \frac{d\theta_z}{dx_i} u + \sum_k \frac{d\theta_{p_k}}{dx_i} v_k + \theta_z v_i,$$

$$\frac{de_{ij}}{dt} = \sum_k \left(\left(\frac{d^3\theta}{dx_i dx_j dx_k} \right) \right) a_k + \left(\frac{d^2\theta_z}{dx_i dx_j} \right) u$$

$$+ \sum_k \left(\frac{d^2\theta_{p_k}}{dx_i dx_j} \right) v_k + \frac{d\theta_z}{dx_j} v_i + \frac{d\theta_z}{dx_i} v_j + \sum_{\alpha\beta} \frac{\partial}{\partial p_{\alpha\beta}} \left(\frac{d^2\theta}{dx_i dx_j} \right) e_{\alpha\beta}.$$

If F is an invariant, the equation which it satisfies is similar to the one given in the previous section, but it contains the additional terms

$$
\begin{aligned}
& -\sum_i F_{a_i}\left\{\sum_k \frac{d\theta_{p_i}}{dx_k} a_k + \frac{\partial\theta_{p_i}}{\partial z} u + \sum_k \frac{\partial^2\theta}{\partial p_i \partial p_k} v_k\right\} \\
& + F_u \theta_z u + F_{v_i}\left\{\frac{d\theta_z}{dx_i} u + \theta_z v_i + \sum_k \frac{d\theta_{p_k}}{dx_i} v_k\right\} \\
+ \sum_{ij}\Bigg\{ & \sum_k \left(\left(\frac{d^3\theta}{dx_i dx_j dx_k}\right)\right) a_k + \left(\frac{d^2\theta_z}{dx_i dx_j}\right) u \\
& + \sum_k \left(\frac{d^2\theta_{p_k}}{dx_i dx_j}\right) v_k + \frac{d\theta_z}{dx_j} v_i + \frac{d\theta_z}{dx_i} v_j + \sum_{\alpha\beta} \frac{\partial}{\partial p_{\alpha\beta}}\left(\frac{d^2\theta}{dx_i dx_j}\right) c_{\alpha\beta}\Bigg\} F_{c_{ij}}.
\end{aligned}
$$

The equations for F are now

$$
\mathbf{S}[P_{jk}F_{A_i} + P_{ki}F_{A_j} + P_{ij}F_{A_k}] - a_k F_{c_{ij}} - a_i F_{c_{jk}} - a_j F_{c_{ik}} = 0 \qquad (i \neq j,\ j \neq k,\ k \neq i), \tag{27}
$$

$$
\mathbf{S}[P_{ij}F_{A_i} + P_{ii}F_{A_j}] - a_j F_{c_{ii}} - a_i F_{c_{ij}} = 0, \tag{28}
$$

$$
\mathbf{S}P_{ij}F_Z - uF_{c_{ij}} = 0, \tag{29}
$$

$$
\mathbf{S}P_{ij}F_{P_k} - v_k F_{c_{ij}} = 0, \tag{30}
$$

$$
\begin{aligned}
\mathbf{S}\left[A_i F_{A_i} - P_i F_{P_i} - P_{ii}F_{P_{ii}} - \sum_{j=1}^n P_{ij}F_{P_{ij}}\right] + \mu F & \\
- a_i F_{a_i} + v_i F_{v_i} + c_{ii}F_{c_{ii}} + \sum_{j=1}^m c_{ij}F_{c_{ij}} &= 0,
\end{aligned} \tag{31}
$$

$$
\begin{aligned}
\mathbf{S}\left[A_i F_{A_j} - P_j F_{P_i} - \sum_{k=1}^n P_{kj}F_{P_{ki}} - P_{jj}F_{P_{ji}}\right] & \\
- a_j F_{a_i} + v_i F_{v_j} + c_{ij}F_{c_{jj}} + \sum_{k=1}^m c_{ki}F_{c_{kj}} &= 0,
\end{aligned} \tag{32}
$$

$$
\mathbf{S}[P_i F_Z] - uF_{v_i} = 0, \tag{33}
$$

$$
\mathbf{S}A_i F_Z - uF_{a_i} = 0, \tag{34}
$$

$$
\mathbf{S}[A_j F_{P_i} + A_i F_{P_j}] - v_j F_{a_i} - v_i F_{a_j} = 0, \tag{35}
$$

$$
\begin{aligned}
\mathbf{S}\left[ZF_Z + \sum_{i=1}^n P_i F_{P_i} + \sum_{\alpha\beta} P_{\alpha\beta}F_{P_{\alpha\beta}}\right] - \left(\sum_{\alpha\beta} c_{\alpha\beta}F_{c_{\alpha\beta}} + uF_u + \sum_{i=1}^n v_i F_{v_i}\right) & \\
= \lambda F \qquad (i, j, k, \alpha, \beta, = 1, \cdots, n). &
\end{aligned} \tag{36}
$$

We first consider the particular case when all the original expressions f are of the first order.

In this case $P_{ij} = 0\ (i, j = 1, \cdots, n)$, and it may be readily seen that F does not in general involve any of the magnitudes $c_{\alpha\beta}$.

We have therefore the reduced system of equations for F:

$$(37)\qquad \mathbf{S}\,[A_i F_{A_i} - P_i F_{P_i}] + uF - a_i F_{a_i} + v_i F_{v_i} = 0,$$

$$(38)\qquad \mathbf{S}\,[A_i F_{A_j} - P_j F_{P_i}] + v_i F_{v_j} - a_j F_{a_i} = 0,$$

$$(39)\qquad \mathbf{S} P_i F_Z - uF_{v_i} = 0,$$

$$(40)\qquad \mathbf{S} A_i F_Z - uF_{a_i} = 0,$$

$$(41)\qquad \mathbf{S}\,[A_j F_{P_i} + A_i F_{P_j}] - v_j F_{a_i} - v_i F_{a_j} = 0,$$

$$(42)\qquad \mathbf{S}\left[ZF_Z + \sum_{i=1}^{n} P_i F_{P_i}\right] - uF_u - \sum_{i=1}^{n} v_i F_{v_i} = \lambda F \qquad (i, j=1, \cdots, n).$$

From (39) and (40) we deduce that F must be a function of u, W, P_i, A_i, where

$$W = uZ + \sum a_i A_i + \sum v_i P_i;$$

and the equations for F are now

$$(43)\qquad \mathbf{S}\,[A_i F_{A_i} - P_i F_{P_i}] + \mu F = 0,$$

$$(44)\qquad \mathbf{S}\,[A_i F_{A_j} - P_j F_{P_i}] = 0,$$

$$(45)\qquad \mathbf{S}\,[A_j F_{P_i} + A_i F_{P_j}] = 0,$$

$$(46)\qquad \mathbf{S}\left[\sum_{i=1}^{n} P_i F_{P_i}\right] + uF_u = \lambda F \qquad (i, j, = 1, \cdots, n).$$

Hence the quantities W are absolute invariants, and in addition there are the invariants given in the previous section. Also u is an invariant.

The additional invariants obtained are readily interpreted. The function

$$u \equiv dz - \sum p_i dx_i,$$

is an invariant arising in connection with the contact transformation itself, and W may easily be shown to be

$$df \equiv \frac{\partial f}{\partial z} dz + \sum_i \frac{\partial f}{\partial x_i} dx_i + \sum_i \frac{\partial f}{\partial p_i} dp_i.$$

Suppose that $u = 0$. Then unless the r-row determinants of the matrix

$$\left\| \begin{matrix} A_{1,1} & \cdots & A_{1,n} & P_{1,1} & \cdots & P_{1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ A_{r,1} & & A_{r,n} & P_{r,n} & \cdots & P_{r,n} \end{matrix} \right\|$$

are all zero, and F_Z is zero.

If we write $v_i = A_{r+1,\,i}$ and $a_i = -P_{r+1,\,i}$, the set of equations becomes the same as that given previously in which there were $(r+1)$ expressions and in

which the variables dz, dx_i, dp_i, did not occur, the last equation only being different. The additional solutions are therefore $[f_i, f_{r+1}]$ $(i=1, \cdots, r)$, which are $df_1, df_2, \cdots, df_r$ subject to the condition that $u=0$.

We now consider expressions F of the second order. Suppose first that $u \neq 0$ and that $v_1, \cdots, v_n$ are not all zero.

Write L for

$$uZ + \sum_i a_i A_i + \sum_i v_i P_i + \sum_{\alpha\beta} c_{\alpha\beta} P_{\alpha\beta},$$

then F is readily shown to be a function of the variables u, L, v_i, P_{ij}, A_i $(i, j = 1, \cdots, n)$, which satisfies the system of equations

$$\mathbf{S}\,[P_{jk} F_{A_i} + P_{ki} F_{A_j} + P_{ij} F_{A_k}] = 0 \quad (j \neq k,\, k \neq i,\, i \neq j),$$

$$\mathbf{S}\,[P_{ij} F_{A_i} + P_{ii} F_{A_j}] = 0,$$

$$\mathbf{S}\left[A_i F_{A_i} - P_{ii} F_{P_{ii}} - \sum_{j=1}^{n} P_{ij} F_{P_{ij}}\right] + \mu F + v_i F_{v_i} = 0,$$

$$\mathbf{S}\left[A_i F_{A_j} - P_{jj} F_{P_{ji}} - \sum_{k=1}^{n} P_{kj} F_{P_{ki}}\right] + v_i F_{v_j} = 0,$$

$$\mathbf{S}\left(\sum_{\alpha\beta} P_{\alpha\beta} F_{P_{\alpha\beta}}\right) + uF_u + \sum_{i=1}^{n} v_i F_{v_i} = \lambda F.$$

The magnitudes L are therefore invariants. In addition u is an invariant, and the remaining invariants are solutions of the given system.

L is readily shown to be

$$df \equiv \sum_i \frac{\partial f}{\partial x_i} dx_i + \frac{\partial f}{\partial z} dz + \sum_i \frac{\partial f}{\partial p_i} dp_i + \sum_{\alpha\beta} \frac{\partial f}{\partial p_{\alpha\beta}} dp_{\alpha\beta},$$

and it therefore admits of an immediate interpretation. There are also the invariants obtained in the preceding section which do not involve the magnitudes v_i $(i = 1, \cdots, n)$ and the remaining invariants are those solutions of the set of equations last given which involve these magnitudes.

We see that when certain conditions given in the preceding section hold, the quantities F_{A_i} $(i = 1, \cdots, n)$ are all zero.

There remains the system

$$\mathbf{S}\left(P_{ii} F_{P_{ii}} + \sum_{j=1}^{n} P_{ij} F_{P_{ij}}\right) - v_i F_{v_i} = \mu F,$$

$$\mathbf{S}\left(P_{jj} F_{P_{ji}} + \sum_{k=1}^{n} P_{kj} F_{P_{ki}}\right) - v_i F_{v_j} = 0,$$

$$\mathbf{S}\left(\sum_{\alpha\beta} P_{\alpha\beta} F_{P_{\alpha\beta}}\right) + \sum_{i=1}^{n} v_i F_{v_i} = \lambda F \quad (i, j, \alpha, \beta, = 1, \cdots, n).$$

The solution of these equations is a function of the algebraic invariants and covariants of the r quadratic forms

$$\sum_{ij} P_{ij} v_i v_j .$$

In addition, this function must be homogeneous in the variables v.

These forms

$$\sum_{ij} P_{ij} v_i v_j$$

are those which have been referred to earlier as the Characteristic Invariants of the expressions f.

Now suppose that u and the quantities v are all zero.

In this case F_Z and F_{P_i} $(i = 1, \cdots, n)$ are all zero unless all the r-row determinants of the matrix

$$\left| \begin{matrix} P_{1,11} & \cdots & P_{1,ij} & \cdots & P_{1,nn} \\ \vdots & & \vdots & & \vdots \\ P_{r,11} & \cdots & P_{r,ij} & \cdots & P_{r,nn} \end{matrix} \right|$$

vanish.

We assume that these conditions are not satisfied, and therefore F is a function of a_i, A_i, P_{ij}, c_{ij} $(i, j = 1, \cdots, n)$.

The equations of types (27) and (28) are satisfied by the magnitudes

$$L \equiv \sum_i A_i a_i + \sum_{jk} P_{jk} c_{jk} .$$

It may readily be shown that these are the only independent solutions of this set of equations unless it is possible to make

$$\sum_{i=1}^{r} S_i B_i + AK = 0$$

identically, where any n variables $y_1, y_2, \cdots, y_n$ are taken, and the magnitudes B_i are linear functions of these variables, K is a quadratic function of them, and

$$S_i \equiv \sum_{\alpha\beta} P_{i,\alpha\beta} y_\alpha y_\beta ,$$

$$A \equiv \sum_k a_k y_k .$$

This equation can be satisfied by making the linear functions all equal to A, multiplied by certain constant factors. This leads to solutions of our system of the type L.

If we assume that there is no other way of making the expression considered an identity, we see that the quantities L are the only invariants involving the variables $A_k (k = 1, \cdots, n)$.

It is readily seen that these expressions L are equivalent to the expressions df, subject to the conditions

$$u = 0, \qquad v_k = 0 \qquad (k = 1, 2, \cdots, n).$$

It is easy to show that the remaining solutions of the system are the algebraic invariants of the system of quadratic forms S_i and of the linear form A.

These forms S are again the characteristic invariants of the expressions f; they have however, in this case, the magnitudes y for independent variables.

Combining our results we have the following theorem:

All invariants, of the restricted type considered of r second order expressions f in one dependent and n independent variables are

(1) *Expressions of the type df.*

(2) *Algebraic invariants and covariants of the quadratic forms*

$$\sum_{ij} \frac{\partial f}{\partial p_{ij}} v_i v_j,$$

where the v's are the variables, and

$$v_k = dp_k - \sum_{i=1}^{n} p_{ki} dx_i,$$

provided

$$dz \neq \sum_{i}^{n} p_i dx_i,$$

and the v's are not zero.

(3) *Algebraic invariants of the quadratic forms*

$$\sum_{ij} \frac{\partial f}{\partial p_{ij}} y_i y_j,$$

and of the linear form

$$\sum_{i=1}^{n} y_i dx_i,$$

where the y's are the variables, provided that

$$dz = \sum_{i=1}^{n} p_i dx_i, \qquad dp_i = \sum_{k=1}^{n} p_{ik} dx_k.$$

In the above there are two sets of restrictions on the expressions f.

(1) The r-row determinants must not all be zero in the matrix

$$\begin{matrix} P_{1,11} & \cdots & P_{1,ij} & \cdots & P_{1,nn} \\ \vdots & & & & \\ P_{r,11} & \cdots & P_{r,ij} & \cdots & P_{r,nn} \end{matrix}.$$

(2) It must not be possible to satisfy the identity

$$\sum_{i=1}^{r} S_i B_i + AK \equiv 0,$$

where S, B, A, K, are as previously defined, except by making the B's all constant multiples of A.

The case when the second of the above restrictions is removed requires further consideration. It is obvious that it must be removed if r is great enough, but the whole question will be left for future consideration. At present we content ourselves with a discussion of the case when n is two.

For one expression f, $SB + AK \equiv 0$, provided that S admits A as a factor, since we are neglecting the possibility of B being λA, where λ is a constant.

This gives $P_{11} a_2^2 - P_{12} a_1 a_2 + P_{22} a_1^2 = 0$, as the condition for the existence of further integrals.

If this condition is satisfied, the equations of type (27) and (28) possess the two solutions

$$\alpha_1 = A_1 a_1 a_2 + P_{22} a_1 c_{12} + P_{11} a_2 c_{11},$$

$$\alpha_2 = A_2 a_1 a_2 + P_{11} a_2 c_{12} + P_{22} a_1 c_{11},$$

instead of the single one already given.

It may readily be shown that there are no new invariants, but if

$$I \equiv P_{11} a_2^2 - P_{12} a_1 a_2 + P_{22} a_1^2 = 0,$$

then the two equations

$$\alpha_1 \equiv A_1 a_1 a_2 + P_{22} a_1 c_{12} + P_{11} a_2 c_{11} = 0,$$

$$\alpha_2 = A_2 a_1 a_2 + P_{11} a_2 c_{12} + P_{22} a_1 c_{22} = 0$$

are an invariant system.

We observe that I is an invariant.

The equations $I = 0$, $\alpha_1 = 0$, $\alpha_2 = 0$, taken in conjunction with

$$dz = p_1 dx_1 + p_2 dx_2, \qquad dp_1 = p_{11} dx_1 + p_{12} dx_2, \qquad dp_2 = p_{12} dx_1 + p_{22} dx_2,$$

have an immediate and important interpretation in connection with the differential equation $f = 0$. They are precisely the equations for the characteristics of this differential equation.*

Now suppose that there are two expressions f. In this case there are four equations in seven variables of the types (27) and (28). They therefore possess in general three integrals. Two of these are already known, and are df_1, df_2.

The remaining one may be expressed as the determinant

$$\begin{vmatrix} \lambda_{1,1}, & \lambda_{1,2}, & 0 \\ K_{1,1}, & K_{1,2}, & I_1 \\ K_{1,2}, & K_{2,2}, & I_2 \end{vmatrix} \equiv \Delta,$$

* GOURSAT, *Equations aux dérivées partielles du second ordre*, vol. 2, p. 174.

where

$$\lambda_1 = 2P_{11}a_2 - P_{12}a_1,$$
$$\lambda_2 = 2P_{22}a_1 - P_{12}a_2,$$
$$K_1 = a_1a_2A_2 + P_{11}a_2c_{12} + P_{22}a_1c_{22},$$
$$K_2 = a_1a_2A_1 + P_{22}a_1c_{12} + P_{11}a_2c_{11},$$
$$I = P_{22}a_1^2 - P_{12}a_1a_2 + P_{11}a_2^2.$$

It may easily be shown that Δ/a_1a_2 satisfies the remaining system of equations, and therefore the complete system of integrals is df_1, df_2, Δ/a_1a_2, and the invariants of the binary forms

$$(P_{1,11}, \tfrac{1}{2}P_{1,12}, P_{1,22} \between \ast)^2, \qquad (P_{2,11}, \tfrac{1}{2}P_{2,12}, P_{2,22} \between \ast)^2, \qquad (a_1, a_2 \between \ast).$$

We omit for the present the interpretation of Δ, and proceed to consider the case in which there are three expressions f.

In this case the equations of types (27) and (28) have the five functionally independent integrals df_1, df_2, df_3 and

$$H \equiv \begin{vmatrix} P_{1,11} & A_{1,1} & P_{1,22} \\ P_{2,11} & A_{2,1} & P_{2,22} \\ P_{3,11} & A_{3,1} & P_{3,22} \end{vmatrix} - \begin{vmatrix} A_{1,2} & P_{1,12} & P_{1,22} \\ A_{2,2} & P_{2,12} & P_{2,22} \\ A_{3,2} & P_{3,12} & P_{3,22} \end{vmatrix},$$

$$K \equiv \begin{vmatrix} P_{1,11} & A_{1,2} & P_{1,22} \\ P_{2,11} & A_{2,2} & P_{2,22} \\ P_{3,11} & A_{3,2} & P_{3,22} \end{vmatrix} - \begin{vmatrix} P_{1,11} & P_{1,12} & A_{1,1} \\ P_{2,11} & P_{2,12} & A_{2,1} \\ P_{3,11} & P_{3,12} & A_{3,1} \end{vmatrix}.$$

We substitute these integrals in the remaining equations, and the complete system of independent integrals of the equations thus obtained is the system of invariants of the binary forms

$$(P_{1,11}, \tfrac{1}{2}P_{1,12}, P_{1,22} \between \ast)^2,$$
$$(P_{2,11}, \tfrac{1}{2}P_{2,12}, P_{2,22} \between \ast)^2,$$
$$(P_{3,11}, \tfrac{1}{2}P_{3,12}, P_{3,22} \between \ast)^2,$$
$$(K, H \between \ast),$$
$$(a_1, a_2 \between \ast).$$

If we take the variables to be $a_2, -a_1$, we see that the solutions in question are the invariants and covariants of the binary forms

$$Ka_2 - Ha_1,$$
$$P_{1,11}a_2^2 - P_{1,12}a_1a_2 + P_{1,22}a_1^2,$$
$$P_{2,11}a_2^2 - P_{2,12}a_1a_2 + P_{2,22}a_1^2,$$
$$P_{3,11}a_2^2 - P_{3,12}a_1a_2 + P_{3,22}a_1^2.$$

In addition to these there are the three solutions df_1, df_2, df_3, and these are all the functionally independent integrals of our type.

§ 4.

We shall consider to a small extent the more general type of expression f, that is to say the type in which there are more dependent variables than one.

Let there be m dependent variables, $z_1, \cdots, z_m$, and as before let there be n independent variables, $x_1, \cdots, x_n$. We use the same notation as on pp. 288, 289, and in addition $df/dx_\mu = A_\mu$.

In this case, the most general contact transformation possible may easily be shown to be a point transformation.

Let the expressions f_λ be of the first order, that is to say, let them be functions of the variables x_k, z_i, p^i_k $(i = 1, 2, \cdots, m;\ k = 1, 2, \cdots, n)$.

If F is any first order invariant we have

$$\frac{dF}{dt} = \left(\mu_0 \frac{d\Omega_0}{dt} + \mu_1 \frac{d\Omega_1}{dt} \right) F,$$

where

$$\frac{d\Omega_0}{dt} = \sum_{k=1}^{n} \frac{\partial \xi_k}{\partial x_k} + \sum_{i=1}^{m} \frac{\partial \zeta_i}{\partial z_i},$$

$$\frac{d\Omega_1}{dt} = \sum_{i=1}^{m} \sum_{k=1}^{n} \frac{\partial \pi^i_k}{\partial p^i_k},$$

and μ_0, μ_1, have the meanings given on p. 288.

Expanding dF/dt and equating to zero the coefficients of ξ_k, ζ_i, $\partial\zeta_i/\partial x'_\mu$, we see that the variables x, z, do not occur explicitly, and that the variables p, x, only occur through the variables A.

We assume F to be a function of the variables A, Z, P, and

$$a_k \equiv dx_k, \quad v_i \equiv dz_i - \sum_{k=1}^{n} p^i_k dx_k \qquad (k = 1, \cdots, n,\ i = 1, \cdots, m).$$

The values of the various increments involved are the following:

$$\frac{d}{dt} a_k = \sum_{\lambda=1}^{m} \frac{\partial \xi_k}{\partial z_\lambda} v_\lambda + \sum_{\mu=1}^{n} \frac{d\xi_k}{dx_\mu} a_\mu,$$

$$\frac{d}{dt} v_i = \sum_{\lambda=1}^{m} \frac{\partial \theta_i}{\partial z_\lambda} v_\lambda,$$

$$-\frac{dA_k}{dt} = \sum_{r=1}^{n} \frac{d\xi_r}{dx_k} A_r + \sum_{i=1}^{m} \sum_{\mu=1}^{n} P^i_\mu \frac{d^2\theta_i}{dx_\mu\, dx_k},$$

$$-\frac{dZ_\lambda}{dt} = \sum_{r=1}^{n} \frac{\partial \xi_r}{\partial z_\lambda} A_r + \sum_{s=1}^{m} \frac{\partial \theta_s}{\partial z_\lambda} Z_s + \sum_{i=1}^{m} \sum_{k=1}^{n} P^i_k \frac{d}{dx_k} \left(\frac{\partial \theta_i}{\partial z_\lambda} \right),$$

$$-\frac{dP^h_k}{dt} = \sum_{i=1}^{m} P^i_k \frac{\partial \theta_i}{\partial z_h} - \sum_{\sigma=1}^{n} P^h_\sigma \frac{d\xi_k}{dx_\sigma} \qquad (h, i = 1, 2, \cdots, m;\ k = 1, 2, \cdots, n).$$

Also

$$\mu_0 \frac{d\Omega_0}{dt} + \mu_1 \frac{d\Omega_1}{dt} = \mu \sum_{k=1}^{n} \frac{d\xi_k}{dx_k} + \nu \sum_{h=1}^{m} \frac{\partial \theta_h}{\partial z_h},$$

where μ and ν are constants depending on μ_0 and μ_1.

We substitute these values in the equation for F and equate the coefficients of the various derivatives of ξ and ζ to zero. We thus obtain the following system of equations for F:

(47) $$a_\sigma F_{a_k} - \mathbf{S} A_k F_{A_\sigma} + \mathbf{S} \sum_{i=1}^{m} P^i_\sigma F_{P^i_k} = 0 \qquad (\sigma \neq k),$$

(48) $$a_k F_{a_k} - \mathbf{S} A_k F_{A_k} + \mathbf{S} \sum_{i=1}^{m} P^i_k F_{P^i_k} = \mu F,$$

(49) $$v_\lambda F_{a_k} - \mathbf{S} A_k F_{Z_\lambda} = 0,$$

(50) $$v_\lambda F_{v_i} - \mathbf{S} Z_i F_{Z_\lambda} - \mathbf{S} \sum_{k=1}^{n} P^i_k F_{P^\lambda_k} = 0 \qquad (\lambda \neq i),$$

(51) $$v_i F_{v_i} - \mathbf{S} Z_i F_{Z_i} - \mathbf{S} \sum_{k=1}^{n} P^i_k F_{P^i_k} = \nu F,$$

(52) $$-\mathbf{S} P^i_\mu F_{A_k} - \mathbf{S} P^i_k F_{A_\mu} = 0,$$

(53) $$-\mathbf{S} P^i_k F_{Z_\lambda} = 0$$

$$(\sigma, k, \mu = 1, \cdots, n);\ (i, \lambda = 1, \cdots, m).$$

The equations (49) give solutions of type

$$\Delta \equiv \sum_{k=1}^{n} a_k A_k + \sum_{i=1}^{m} v_i Z_i,$$

and the equations (53) then show that the Δ's cannot enter into F unless the number r of expressions f_λ is greater than mn.

From equations (52) we deduce that the variables A do not occur in F unless

$$r > \tfrac{1}{2} m (n + 1).$$

In the case when m is unity, the equations are not all independent, and r need not satisfy this last condition.

Suppose that r is less than this number, then F is a function of the variables $v_1, \cdots v_m, P^1_1, \cdots P^m_n$, which satisfies the equations

$$\mathbf{S} \sum_{i=1}^{m} P^i_\sigma F_{P^i_k} = 0, \qquad (\sigma \neq k),$$

$$\mathbf{S}\sum_{i=1}^{m} P^i_k F_{P^i_k} = \mu F,$$

$$\mathbf{S}\sum_{k=1}^{n} P^i_k F_{P^\lambda_k} - v_\lambda F_{v_i} = 0,$$

$$\mathbf{S}\sum_{k=1}^{n} P^i_k F_{P^i_k} - v_i F_{v_i} = -\nu F.$$

The last two equations show that F must be an invariant or covariant of the linear forms

$$\sum_{i=1}^{m} P^i_{\lambda,k} v_i \qquad (k=1, 2, \cdots, n;\ \lambda=1, 2, \cdots, r).$$

The two first equations show that F must at the same time be an invariant of the linear forms

$$\sum_{k=1}^{n} P^i_{\lambda,k} y_i \qquad (i=1, 2, \cdots, m;\ \lambda=1, 2, \cdots, r)$$

Bryn Mawr College.

THE ELEMENTARY TREATMENT OF CONICS BY MEANS OF THE REGULUS.

BY PROFESSOR CHARLOTTE ANGAS SCOTT.

(Read before the American Mathematical Society, February 25, 1905.)

INASMUCH as projective plane geometry depends on properties of space (since the theorem on triangles in perspective in a plane cannot be established by plane geometry without the use of metric concepts), there is no logical objection to the employment of space considerations in proving theorems of plane geometry. The regulus supplies extremely simple proofs of the properties of a conic, whether this be considered as a locus or an envelope; these proofs have the advantage of connecting the points of a conic and the tangents of a conic from the first, instead of leaving the connection to be proved later by an elaborate chain of reasoning. The method makes unnecessary also the treatment of the tangent as the limiting position of a chord when the determining points become indistinguishable, which is open to some objections in pure geometry; not on account of the assumption as to continuity, for this has already been admitted in the proof of the fundamental theorem of projective geometry, but with regard to elegance and directness of proof. This note contains the application of the method to the proofs of the theorems of Chasles, Brianchon, and Pascal, and of polar and involution properties.

The regulus is the system of lines (rays) that meet three non-incident lines, the directors. By means of the fundamental theorem of projective geometry it is proved that the regulus is crossed by a second regulus; any three rays of either serve as directors of the other; through any point on a ray of either there passes a ray of the other.

Three directors, u, v, w, and three rays a, b, c (that is, two incident triads of lines) give rise to an interesting figure on which the proofs depend. The pairs of lines au, bv, cw determine points A, B, C and planes α, β, γ; the three planes meet in a point O, the three points lie in a plane ω. The point $\begin{pmatrix} a & b & c \\ u & v & w \end{pmatrix}$ and the plane $\begin{pmatrix} a & b & c \\ u & v & w \end{pmatrix}$, that is, O and ω, are pole and polar. The pairs of points (bw, cv), (cu, aw), (av, bu), that is, $\underset{v\ \ w}{\overset{b\ \ c}{\times}}$, $\underset{w\ \ u}{\overset{c\ \ a}{\times}}$, $\underset{u\ \ v}{\overset{a\ \ b}{\times}}$, lie on lines through O, namely, the lines $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$; the pairs of planes $\underset{v\ \ w}{\overset{b\ \ c}{\times}}$, $\underset{w\ \ u}{\overset{c\ \ a}{\times}}$, $\underset{u\ \ v}{\overset{a\ \ b}{\times}}$ meet in lines in ω, namely, BC, CA, AB. The points are named as shown in Fig. 1, the planes are named to correspond, *e. g.*, the point av is A', the plane av is α'; the planes α', β', γ' determine the point O', etc.

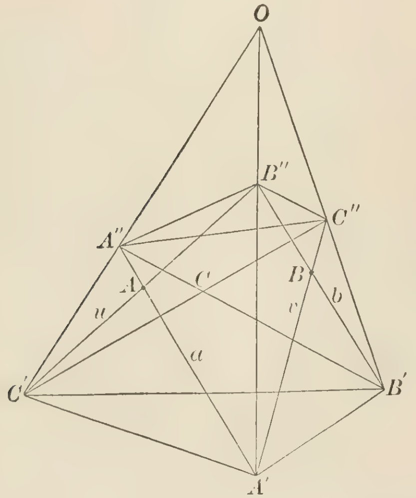

FIG. 1.

The triangles $A'B'C'$, $B''C''A''$ are perspective from O; $A''B''C''$, BCA from O'; ABC, $B'C'A'$ from O''. It is easily shown that O, O', O'' are collinear, ω, ω', ω'' are coaxial; and that if the rays and directors are associated in the reversed order $\begin{pmatrix} a & c & b \\ u & v & w \end{pmatrix}$, the three new centres of perspective lie on the axis of $\omega\ \omega'\ \omega''$, and the axis of the new planes is $O\ O'\ O''$. These facts, however, are not needed for the present purpose.

It is at once seen that O, ω are harmonic to the pairs of lines au, bv, cw, that is, to the two triads of rays a, b, c and directors u, v, w, hence to the complete system of transversals to a, b, c (directors) and transversals to u, v, w (rays). This shows that the rays and directors intersect in pairs at points on ω, lie in pairs (the same pairs) on planes through O; the two of a pair d, x thus associated are harmonic with regard to O and ω. Any

chord through O joins a point on a ray d to a point on the associated director x, and is therefore divided harmonically by O, ω. Moreover, if O be any point on a chord that meets a ray d and director x, then the point dx lies on the polar plane of O.

Any line p, not a director, that meets one ray meets one other, but no more. For let p pass through A (*i. e.*, au); the plane pu meets v, w, at V, W; the line VW is therefore a ray, and is met by p. The only possible exception arises when VW is itself the ray a; then the line p meets the regulus at the point A only (or, the line p meets only one ray of the regulus); p is a tangent line, and the plane α (*i. e.*, au), in which lie all tangent lines through the point A (*i. e.*, au), is a tangent plane. The facts proved above as to the relation of O, ω may be stated in the form: The points of contact of tangent planes from the point $\begin{pmatrix} a & b & c \\ u & v & w \end{pmatrix}$ lie on the plane $\begin{pmatrix} a & b & c \\ u & v & w \end{pmatrix}$.

In order that a tangent plane may contain a line q, it must contain any two points S, S' on the line; hence the point of contact must lie on both σ and σ', that is on a line q'; q, q' are conjugate lines. Hence if there is one tangent plane through q, there is precisely one other, unless q' is itself a ray or director, which happens only when q is a ray or director, and then q' coincides with q.

The ranges of points determined on two directors u, v by the rays $a, b, c, d, \cdots$ are sections of the axial pencil $x.abcd \cdots$ (where x is any other director) by the transversals u, v; hence they are projective. The regulus is therefore the system of lines that join corresponding points of projective ranges on two non-incident lines. Again, the axial pencils $u.abcd \cdots$, $v.abcd \cdots$ have a common section by the transversal x, hence they are projective. The regulus is therefore the system of lines determined by corresponding planes of two projective axial pencils whose axes are non-incident.

Let the regulus be cut by a plane ω; each ray is thus associated with a particular director; the two, d, x, meet in a point D on ω, and lie in a plane δ (a tangent plane) through O. The projective axial pencils $u.abcd \cdots$, $v.abcd \cdots$ are cut in projective flat pencils, centers A and B; hence the plane section of a regulus (which is a point system of the second order, since a line through A cuts one other ray) is the locus of the intersection of corresponding rays of two projective flat

pencils. Consider also the line system composed of the tangent lines in the plane ω; these are the intersections of the tangent planes au, bv, cw, dx, ey, etc. by ω, they are therefore the projections of rays $a, b, c, d, e, \cdots$ (or of directors $u, v, w, x, y, \cdots$) from O on ω; call them $a', b', c', \cdots$. Since the ranges $u.abcd\cdots$, $v.abcd\cdots$ are projective, their projections from O on ω are projective. Hence the lines $a', b', c', d', \cdots$ in the plane ω connect the corresponding points of projective ranges on the two tangents a', b'; that is, the line system (which, by what has been said about tangent planes, is seen to be of the second order) is composed of the lines that join corresponding points of projective ranges. Moreover, as to the relation of the two systems: the flat pencil $A.ABCD\cdots$ in the plane $\omega =$ the axial pencil $u.abcd\cdots =$ range $v.abcd\cdots =$ range $v'.a'b'c'd'\cdots$ in the plane ω. Thus the section (aggregate of points and lines) has the property that the pencil subtended at any point A by the points C, D, E, F is projective with the range determined on any tangent b' by the tangents c', d', e', f', which is Chasles's theorem. The section will now be called a conic.

Since the pencil determined by four points is projectively

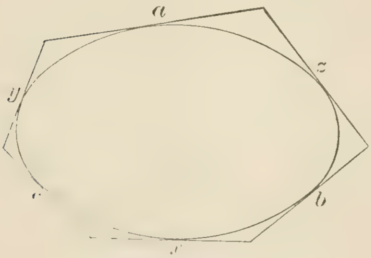

FIG. 2.

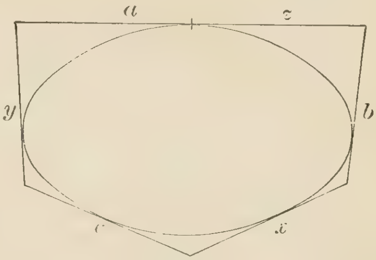

FIG. 3.

the same wherever on the section its vertex may lie, and similarly for the range determined by four tangents, it is unnecessary to specify the vertex or base; it is sufficient to speak of the four points or four lines. Since also the range determined by four rays on any director of the regulus depends only on the rays, it is sufficient to speak of the four rays. What has been proved above can be stated in the form: Four points of a conic are projective with the four rays, or four directors, that pass through them; four tangents to a conic are projective with the four rays, or four directors, that pass through their points of contact.

If four rays a, b, c, d are projective with four directors u, v, w, x, their intersections are coplanar. For let y be the director through the point D in which the plane au, bv, cw (that is, ABC) meets d; then the rays a, b, c, d are projective with the points A, B, C, D, and these are projective with the directors u, v, w, y. Hence $u\,v\,w\,x = u\,v\,w\,y$, and y is therefore the same as x.

The polar properties follow at once from the harmonic relation borne to the rays and directors by O, ω. A point T in the plane ω has a polar plane τ, which by harmonic symmetry of the whole figure to $O\omega$ must pass through O. The section of τ by ω is the polar line of T with regard to the section considered, and it has already been shown that any chord through T, and therefore a chord of the section, is harmonically divided by $T\tau$. Hence follows the usual quadrilateral construction for pole and polar.

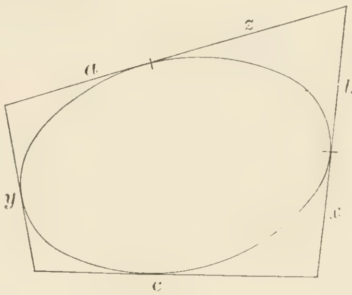

FIG. 4.

To prove Brianchon's theorem, take the six tangents as pro-

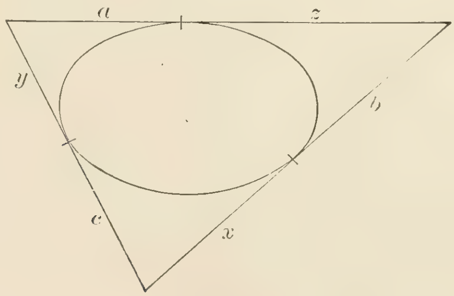

FIG. 5.

jections alternately of rays and directors, as shown in Fig. 2. In space, the lines $\overset{b\ c}{\underset{y\ z}{\times}}$, $\overset{c\ a}{\underset{z\ x}{\times}}$, $\overset{a\ b}{\underset{x\ y}{\times}}$ meet at a point S; projecting the figure from O on to ω, we obtain three concurrent lines joining opposite vertices of the circumscribing hexagon.

To prove the special cases of this theorem for the circum-

scribing pentagon, quadrilateral, triangle, regard the necessary number of tangents (one, two or three) as projections of both a ray and a director, as shown in Figs. 3, 4, 5; the proof then applies without change, the point az being the point of contact of the tangent.

To prove Pascal's theorem, take alternately the rays and directors that pass through the six points, Fig. 6. The common lines of the planes $\overset{b\ c}{\underset{y\ z}{\times}}$, $\overset{c\ a}{\underset{z\ x}{\times}}$, $\overset{a\ b}{\underset{x\ y}{\times}}$ lie in a plane σ; hence in the section by the plane ω the common points of the lines $\overset{b\ c}{\underset{y\ z}{\times}}$, $\overset{c\ a}{\underset{z\ x}{\times}}$, $\overset{a\ b}{\underset{x\ y}{\times}}$ lie in a line, the line $\sigma\omega$. The special cases of this theorem, which arise when sides of the inscribed hexagon (one, two, or three) are replaced by tangents, require no change in the proof. Moreover, the same proof applies when the section by the plane ω consists of a ray and a director, that is, when the theorem to be proved becomes Pappus's theorem, that if the vertices of a hexagon are taken to lie alternately on two intersecting lines, the three intersections of opposite sides are collinear.

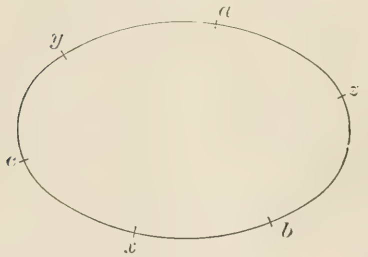

FIG. 6.

It has been mentioned that if a chord meets d, x, the point dx lies on the polar plane of any point on the chord. Hence if chords of a conic, A_1A_2, B_1B_2, C_1C_2, $\cdots$, are concurrent in S, then $A_1B_1C_1D_1 = A_2B_2C_2D_2$; for let a, b, c, d be the rays through A_1, B_1, C_1, D_1, and u, v, w, x the directors through A_2, B_2, C_2, D_2, and let the points au, bv, cw, dx (which are known to lie on σ) be U, V, W, X. Then $A_1B_1C_1D_1 = abcd = UVWX$ (in the section σ) $= uvwx = A_2B_2C_2D_2$. This proof is unaffected by coincidence; D_1D_2 may be the same as A_2A_1. Thus $A_1B_1C_1A_2 = A_2B_2C_2A_1$. Conversely, if in the plane ω $A_1B_1C_1A_2 = A_2B_2C_2A_1$, the chords A_1A_2, B_1B_2, C_1C_2 are concurrent. For since $A_1B_1C_1A_2 = abcd$, and $A_2B_2C_2A_1 = uvwx$, we have $abcd = uvwx$, and therefore by a result obtained earlier, the points au, bv, cw, dx lie on a plane σ. Hence the planes au, bv, cw, dx meet in a point S. But the planes au, dx meet in a line in the plane ω, hence the point S lies in ω. Since then the planes

au, *bv*, *cw*, *dx* meet at a point S in ω, the section of these planes by ω gives chords of the conic concurrent in S. Thus we have the theorem : If points of a conic are projectively paired, they lie on concurrent chords, which is the foundation of the theory of involution on a conic.

For the sake of brevity, as little detail as possible has been given in this note, the design being simply to draw attention to this mode of proving fundamental properties of the conic.

BRYN MAWR COLLEGE,
June, 1905.

AN APPLICATION OF THE THEORY OF DIFFERENTIAL INVARIANTS TO TRIPLY ORTHOGONAL SYSTEMS OF SURFACES.

BY J. E. WRIGHT, M. A.

(Read before the American Mathematical Society, December 29, 1905.)

IT has been proved by Darboux * that a family of surfaces which makes part of a triply orthogonal system must satisfy a differential equation of the third order. This differential equa-

* "Sur les surfaces orthogonales" (*Bulletin de la Société philomath.*, 1866, p. 16), (*Annales de l'Ecole normale*, 1re séries, vol. 3 (1866), p. 97); see Leçons sur les systèmes orthogonaux, pp. 13–14 for complete bibliography.

tion is given by Darboux, in his Leçons sur les systèmes orthogonaux, page 20, in the form $S=0$, where S is a certain six-rowed determinant. Now, since $S=0$ has a geometric interpretation, it is natural to expect that S is a differential invariant. If so, it must be expressible as an algebraic invariant of certain forms which may be readily written down.* In this note S is expressed as an algebraic invariant of the forms in question and certain immediate consequences are given.

The determinant S contains only third and lower derivatives of u with respect to x, y, and z, where $u=$ constant is the family of surfaces considered and x, y, z are rectangular cartesian coördinates. Hence the only possible algebraic forms of which it can be an invariant are

$$\sum_{r=1}^{3}\sum_{s=1}^{3} a_{rs}U_rU_s, \text{ and } S_1,\ S_2,\ S_3,$$

where the notation of the author's paper already quoted is used. Now with the particular set of variables used, namely rectangular cartesians, the first of these forms becomes $U_1^2+U_2^2+U_3^2$, and the others

$$\left(U_1\frac{\partial}{\partial x}+U_2\frac{\partial}{\partial y}+U_3\frac{\partial}{\partial z}\right)^{\lambda} u,$$

when λ takes the values 1, 2 and 3.

It is noticeable that u_{ik} and certain quantities A_{ik} turn up symmetrically in S. Since we have a form $\sum_i\sum_k u_{ik}U_iU_k$, it is suggested that a covariant $\sum_i\sum_k A_{ik}U_iU_k$ is required. Now †

$$A_{ik}=\sum_{l=1}^{3}(u_lu_{ikl}-2u_{il}u_{kl})\ ;$$

we use the ordinary symbolic notation for algebraic invariants and put

$$l_x=S_1,\quad a_x'^2=b_x'=S_2,\quad a_x^3=S_3,$$

$$a_x^2=b_x^2=\sum_{r=1}^{3}\sum_{s=1}^{3}a_{rs}U_rU_s.$$

* See a paper by the author, "The differential invariants of space," *Amer. Jour. of Math.*, vol. 27 (1905), pp. 335-336.

† Darboux, Leçons sur les systèmes orthogonaux, p. 19.

Also let

$$h_x{}^2 = \sum_{i=1}^{3} \sum_{k=1}^{3} A_{ik}\, U_i\, U_k,$$

then it is not difficult to show that

$$h_x^2 = (aab)\,(lab)a_x^2 - 2a'_x b'_x (a'ab)\,(b'ab),$$

provided

$$a_x^2 = U_1^2 + U_2^2 + U_3^2.$$

Hence, since h_x^2 is expressed as a covariant, we have its form for a general a_x^2. The determinant S may now be readily shown to be an invariant of the three quadratics a_x^2, h_x^2 $a_x'^2$ and the one linear form l_x. Its symbolic expression is

$$(hal)\,(ha'l)\,(aa'l).$$

This invariant has a simple geometric interpretation in connection with the ternary forms. Equated to zero it is the condition that the straight line l_x meets the three conics a_x^2, $a_x'^2$, and h_x^2 in six points in involution.

Now that S is expressed as an invariant of the algebraic forms, its expression may be obtained in terms of generalized coordinates ρ_1, ρ_2, ρ_3. The differential invariant theory shows that it is the same algebraic invariant of certain other forms which may be readily obtained. In fact, if $a_x^2 = aU_1^2 + bU_2^2 + cU_3^2 + 2fU_2U_3 + 2gU_3U_1 + 2hU_1U_2$,

$$S_1 = \sum_{i=1}^{3} U_i \frac{\partial u}{\partial \rho_i},$$

$$\Delta S_{m+1} = \begin{vmatrix} \sum_{i=1}^{3} U_i \frac{\partial S_m}{\partial \rho_i}, & F_1, & F_2, & F_3 \\ \frac{\partial S_m}{\partial U_1}, & a, & h, & g \\ \frac{\partial S_m}{\partial U_2}, & h, & b, & f \\ \frac{\partial S_m}{\partial U_3}, & g, & f, & c \end{vmatrix}$$

for $m = 1, 2$.

The expression F_i is equal to

$$\sum_{r=1}^{3}\sum_{s=1}^{3}\begin{bmatrix} r\ s \\ i \end{bmatrix} U_r U_s,$$

where

$$\begin{bmatrix} r\ s \\ i \end{bmatrix} = \tfrac{1}{2}\left\{\frac{\partial a_{ri}}{\partial \rho_s} + \frac{\partial a_{si}}{\partial \rho_r} - \frac{\partial a_{rs}}{\partial \rho_i}\right\}$$

is Christoffel's three index symbol, and Δ is the discriminant of a_x^2.

We may use the generalized expression thus obtained to find the condition that the parametric surfaces $\rho_3 =$ constant may form part of a triply orthogonal system. This condition is however too long to be given here.

As another example we may find the condition that the surfaces $u(x, y, z) =$ constant are all minimal. This condition, in cartesian coordinates, is

$$(u_{11} + u_{22} + u_{33})(u_1^2 + u_2^2 + u_3^2) = u_{11}u_1^2 + u_{22}u_2^2 + u_{33}u_3^2 + 2u_{23}u_2u_3 + 2u_{31}u_3u_1 + 2u_{12}u_1u_2,$$

where suffixes denote differentiations.

Translated into symbolic notation it becomes $(aa'l)^2 = 0$. Hence the condition expresses that a certain straight line cuts two conics harmonically.

If we take our fundamental quadratic form a_x^2 to be $\sum_{i=1}^{3} H_i^2 U_i^2$ and our minimal system $\rho_3 =$ constant, this invariant condition gives $H_1\, \partial H_2/\partial \rho_3 + H_2\, \partial H_1/\partial \rho_3 = 0$, and hence, of a triply orthogonal system, the parametric surfaces $\rho_3 =$ constant are minimal if $H_1 H_2$ is a function of ρ_1 and ρ_2 only, where the element of length is given by

$$ds^2 = H_1^2 d\rho_1^2 + H_2^2 d\rho_2^2 + H_3^2 d\rho_3^2.$$

BRYN MAWR,
January, 1906.

On the Arrangement of the Real Branches of Plane Algebraic Curves.

BY V. RAGSDALE.

Introduction.

In the consideration of any problem relating to the number and arrangement of the real branches of plane algebraic curves, the division of circuits into the two classes *odd* and *even* is of fundamental importance.* An odd circuit can be met by a straight line in an odd number of points only; an even circuit in an even number of points. Two odd circuits have an odd number of intersections; an even and an odd circuit, or two even circuits have an even number of points in common. Hence, as Zeuthen shows (*Sur les differentes formes des courbes planes du quatrième ordre,* Math. Ann. VII, 1873, pp. 410-432), a non-singular curve of even order must be composed entirely of even circuits, and a non-singular curve of odd order must have one circuit odd and the rest even.†

In a paper published in 1876 (*Ueber die Vieltheiligkeit der ebenen algebraischen Curven,* Math. Ann. X, pp. 189-198) Harnack proved that a curve cannot have more than $p+1$ circuits, where p denotes the genus of the curve; also that for every value of p, a curve of some order does exist having $p+1$ real branches. In particular, if p be of the form $\frac{1}{2}(n-1)(n-2)$, there exists a non-singular n^{ic} with $\frac{1}{2}(n-1)(n-2)+1$ real branches. Later, Hilbert (*Ueber*

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* Von Staudt, *Geometrie der Lage,* 1847, p. 80.

† Zeuthen (*loc. cit.* p. 426) proves the existence of a quartic circuit with two double points, which is met by every straight line in at least two real points, and hence can not be projected into the finite part of the plane. Cayley (*On Quartic Curves,* Collected Papers, V. op. 361, 1865) points out that the sextic may be composed of one non-singular circuit which is met by every straight line in at least two real points. And C. A. Scott (*On the Circuits of Plane Curves,* Transactions of the American Mathematical Society, 1902) establishes the general theorem as to the existence of circuits that cannot be projected into the finite part of the plane. In the following pages, however, the only circuits that present themselves are those which can be projected into the finite, and for these the term *oval* is here employed.

die reellen Züge algebraischer Curven, Math. Ann. XXXVIII, 1890, pp. 115-138) considered certain possibilities of arrangement for the circuits of a non-singular n^{ic} when the maximum number of branches is present, and Hulburt (*A Class of New Theorems on the Number and Arrangement of the Real Branches of Plane Algebraic Curves*, American Journal of Mathematics, XIV, 1892, pp. 246-250) extended Hilbert's theorems to certain cases of curves with double points. Hilbert proved that for n even, not more than $\frac{1}{2}(n-2)$ of the $p+1$ ovals can be nested; that is, so situated that the first lies *inside** a second, the second inside a third, and so on; and that curves of even order do exist having $p+1$ ovals, $\frac{1}{2}(n-2)$ of which are nested; similarly, that for n odd, not more than $\frac{1}{2}(n-3)$ ovals can be nested, if the maximum number of circuits is present, and that curves of odd order do exist having $p+1$ circuits, $\frac{1}{2}(n-3)$ of which are nested ovals.

A footnote to this paper† contains the statement that if the non-singular sextic have its maximum number of branches, eleven, these cannot all lie external to one another. Hilbert speaks of the process by which he arrived at this conclusion as "ausserordentlich umständlich," but no hint as to the character of the argument is given, and no proof of the statement has ever been published. However, if such a limitation on the arrangement of the ovals does exist for the 6^{ic}, there arises at once the question as to the existence of a similar limitation for all non-singular curves with the maximum number of branches. For curves of odd order no such restriction holds,—at least, in the form stated by Hilbert,—for it can be shown that a non-singular curve of odd order may have the maximum number of circuits with every oval lying outside the others. For the discussion of the question for curves of even order, however, it is convenient to cast Hilbert's statement into a slightly different form.

For the two types of the 6^{ic} given by Hilbert, the only types that can be derived by his method of generation, the arrangement of the ovals is the following:

(1) An oval O; 1 oval inside O, and 9 outside.

(2) An oval O; 9 ovals inside O, and 1 outside.

It is seen that the numbers of ovals "inside" and "outside" are interchanged in the two cases, and the natural inference is, that the law of arrangement to

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*For the definition of "inside" and "outside" of a closed circuit see Von Staudt (*l. c.* p. 90) and Zeuthen (*l. c.* p. 410).

† l. c. pp. 118–119.

which the ovals are subject, is independent of the distinction between the "inside" and "outside" of a closed circuit, as defined by Von Staudt and Zeuthen, and, in fact, is based on no distinctive or permanent property of any one region of the plane. Thus the division of the plane by the curve $u = 0$ into regions where u is positive and regions where u is negative offers a more promising basis for investigation of the problem, because of the element of arbitrariness introduced in ascribing to a certain region the positive rather than the negative sign. Suppose that the curve is non-singular and of even order, and that all its ovals have been projected into the finite. According to the usual convention let the sign be determined so that the expression u is positive at infinity. A region where u is negative may be a region bounded by a single circuit as in Fig. 1, or a region bounded by two or more circuits as in Fig. 2.

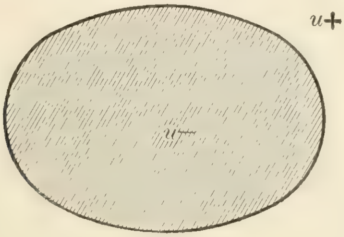

Fig. 1.

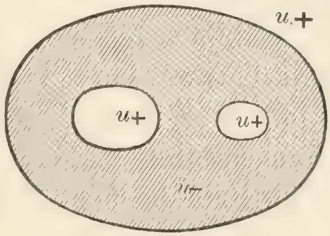

Fig. 2.

Each additional boundary introduces a new positive region. If such a boundary, or an oval which cuts off in the midst of a region where u is negative a region in which u is positive, be called an *internal* oval, and an oval which cuts off in the midst of a region where u is positive a region in which u is negative, an *external* oval, Hilbert's statement can be expressed as follows: *If the non-singular sextic have its maximum number of branches, at least one of the eleven ovals must be internal*;—that is, not more than ten of the eleven ovals can be external. There is as yet no formal proof forthcoming for this statement in either its original or altered form, but as curves of higher order are investigated a most interesting law governing the arrangement of the ovals presents itself so persistently, and in curves of such widely different types, as to give strong reasons for belief in the existence of a general theorem. It is found that the 8^{ic}, 10^{ic}, 12^{ic}, 14^{ic}, ..., with the maximum number of circuits, will have respectively 3, 6, 10, 15,...., or more internal ovals. And in general, *if the non-singular* $2n^{ic}$ *have the maximum number of branches, at least* $\frac{1}{2}(n-1)(n-2)$ *of the* $p+1$ *ovals must be internal*; *or not more than* $n^2 + \frac{1}{2}(n-1)(n-2)$ *can be external.*

As will be shown later, the only processes by which curves with the maximum number of branches have been derived, yield curves of even order whose circuits conform to this law of arrangement. These are the two processes employed by Harnack and Hilbert. The *Harnack process* offers two modes of generation, each of which determines a distinct law of arrangement for the circuits of the derived curve C_{2n}; but these two laws and all modifications of them which arise from combinations of the two modes of generation differ only in the distribution of the internal ovals. The number in every case is $\frac{1}{2}(n-1)\,(n-2)$. For example, of the 22 ovals of the 8^{ic}, 3 are internal, though these 3 may be distributed in two ways (Fig. $3_{(b),\,(c)}$). Of the 37 ovals of the 10^{ic}, 6 are internal, though these 6 may be distributed in four ways (Fig. $3_{(d),\,(e),\,(f),\,(g)}$).

The *Hilbert process* gives less simple arrangements of the circuits. Hilbert's own statement is, that if the $2n^{ic}$ have the maximum number of nested ovals, $n-1$, the remaining ovals must be external to one another and may be distributed in various ways in the annular regions bounded by two successive nested ovals, and in the region lying outside the nest. It is shown (p. 13) that the simplest arrangement of these remaining ovals is represented by the following scheme, which gives the number of ovals in the annular regions, beginning with the innermost ring, $0, 2, 4, 6, 8, \ldots 2n-10,\ 2n-8,\ 2n-6$; the other ovals lie outside the nest entirely. In this case the number of internal ovals is exactly $\frac{1}{2}(n-1)\,(n-2)$. But for all curves of order $2n\,(2n > 6)$, the process gives choice of three distinct modes of generation, and hence affords various possibilities for the arrangement of the circuits. It is still true, however, that no type of $2n^{ic}$ obtained has less than $\frac{1}{2}(n-1)\,(n-2)$ internal ovals.

Both these processes are based upon the principle of small variation from a special degenerate curve. This reducible curve is composed of an $m-k^{ic}$ with the maximum number of circuits and an auxiliary curve of order k which bears a certain specified relation to the $m-k^{ic}$. The m^{ic} obtained has the maximum number of circuits, and bears a relation to the auxiliary curve similar to that possessed by the $m-k^{ic}$. The two methods differ only in the type of auxiliary curve employed. The Harnack process is characterized by the use of the straight line as auxiliary curve; the Hilbert process by the use of the ellipse. Hulburt has proved* that in the generation of curves by the method of small variation, the only auxiliary curves that will yield the maximum number of

*l. c. p. 250.

branches on the derived m^{ic}, are the straight line and the conic; that is, the only processes of this type that will give curves with the maximum number of circuits are the Harnack and Hilbert methods, and modifications of the Hilbert method due to the use of other conics as the auxilary curve. Hence if all non-singular curves with the maximum number of branches are obtainable by the method of small variation, the law which expresses the arrangement of ovals for curves derived by the Harnack and Hilbert processes becomes a general law, and holds for *all* curves of even order with the maximum number of circuits. But whether the law is of perfect generality or not, it is of interest to investigate more fully the various types of curves that can be derived by these different methods.

Curves with the Maximum Number of Branches Derived by the Harnack and Hilbert Processes of Generation.

Curves Derived by the Harnack Process.—Let C_{n-1} be a non-singular curve of order $n-1$ with the maximum number of circuits, and let a straight line v meet one circuit of the curve in $n-1$ real and distinct points which have the same order of succession on C_{n-1} as on v. Harnack shows then that by a proper choice of δ and the straight lines, $l_1, l_2, l_3, \ldots l_n$, $C_n \equiv v \cdot C_{n-1} + \delta \cdot \prod_{i=1}^{i=n} l_i = 0$ can be made to represent a curve of order n having properties corresponding to those of the $n-1^{ic}$. Certainly for $n=3$, such an $n-1^{ic}$ exists; viz., a conic cut in two real points by a straight line v. Let three lines l_1, l_2, l_3 be chosen so that they cut the infinite segment of v. Then the cubic represented by the equation $C_2 \cdot v + \delta \cdot l_1 l_2 l_3 = 0$ will pass through the intersections of C_2 and v with $l_1 l_2 l_3$, and for a small value of δ will have the maximum number of circuits, two. Moreover, the infinite branch is cut by the straight line v in three real points (Fig. 1, Plate I). The quartic can be derived from the cubic in the same way. Let the lines l_1, l_2, l_3, l_4 be so chosen that they cut the same segment of v; then for a proper choice of δ, the equation, $C_4 \equiv C_3 \cdot v + \delta \cdot \prod_{i=1}^{i=4} l_i = 0$, will represent a quartic with four ovals, one of which meets v in four real points (Fig. 2, Plate I). Similarly, the quintic with the maximum number of circuits can be obtained from the quartic, and the sextic from the quintic, and so on.

The restrictions imposed on the $n-1^{ic}$, viz.: (1) That the $n-1^{ic}$ must have the maximum number of branches, (2) that a straight line v must cut C_{n-1}

in $n - 1$ real and distinct points, (3) that all these points must lie on the same circuit of C_{n-1}, are necessary in order that C_n may have the maximum number of branches. Let that circuit of the $n - 1^{ic}$ which is cut by v in $n - 1$ real points be called the generating circuit g_{n-1}. Restrictions (2) and (3) require that g_{n-1} be the infinite branch, if $n - 1$ is odd. There are also certain restrictions that must be imposed on the lines $l_1, l_2, l_3, \ldots l_n$. The points in which C_n cuts v are determined by the points common to v and the lines $l_1, l_2, l_3, \ldots l_n$. If an odd number of these lines cut any finite segment determined on v by C_{n-1} (*n odd*) or *any* segment (*n even*) the number of circuits of C_n falls short of the maximum number. It is clear that by admitting imaginary lines there can be obtained from the given $n - 1^{ic}$, n^{ics} with the maximum number of branches, and cut by v in $0, 2, 4, \ldots$ or n real points if n is even, or in $1, 3, 5, \ldots n$ real points if n is odd; and also that these points of intersection, if more than two, may lie on different circuits. But for the generation of the $n + 1^{ic}$ with the maximum number of circuits from the n^{ic} all the intersections of C_n and v must be real and lie on the same circuit g_n. Hence the straight lines $l_1, l_2, l_3, \ldots l_n$ must be chosen to cut the same segment of v, and in n real and distinct points. If n is odd, this segment must be the infinite segment; if n is even, the segment may be any one, finite or infinite. The general arrangement of the circuits of the n^{ic} with the maximum number of branches is the same whether the intersections of C_n and v are all real and lie on the same circuit or not. The difference in the two cases manifests itself in the number of branches on the $n + 1^{ic}$ and curves of higher order derived from the n^{ic}. Hence, in considering the different types of n^{ics} with the maximum number of branches, it is necessary to take account only of those cases where the lines $l_1, l_2, \ldots l_n$ are subject to such restrictions as allow the process to be continued.

In the generation of the quintic from the quartic (Fig. 3, Plate 1), one circuit of C_4 must cut v in four points. Of the four segments of g_4, two with their corresponding segments of v give rise to two ovals lying external to one another. Of the other two, one, together with the infinite segment of v, generates the infinite branch of the 5^{ic}; the other, with its corresponding segment of v, produces an oval on the 5^{ic}, which must lie in one of the regions bounded by a segment of v and a segment of the infinite branch of C_5 (Fig. 3, Plate I). Hence the 6^{ic} arising from this 5^{ic} must have one oval lying inside another; the other ovals are external, five representing the five remaining ovals of the 5^{ic}, and four generated by the other segments of v and the infinite branch of C_5 (Fig. 4, Plate I).

Though with reference to the arrangement of the circuits there is only one kind of 6^{ic} obtained, there are with regard to the 6^{ic} two essentially different positions of the lines $l_1, l_2, \ldots . l_6$, which are of importance in the generation of curves of higher order. According as these lines cut that segment of v which, together with one segment of C_5, encloses a region containing an oval, or one of the other segments, the generating oval encloses (a) one other oval (Fig. 4, Plate I), or (b) includes no oval (Fig. 5, Plate I).

From the 6^{ic} of type (a) is derived a 7^{ic} with the maximum number of circuits each oval of which lies external to the others ; from the 6^{ic} of type (b), a 7^{ic} with the maximum number of branches and with two ovals nested. It is seen from Figs. 4, 5, Plate I, that of the six segments of g_6, three lie on one side of v and with segments of v give rise to three ovals on C_7 which are external to one another; but of the other three, two lie in the region bounded by the third and the finite segment $x_1\, x_6$ of v. This third segment of g_6 and the infinite segment $x_6 x_1$ of v give rise to the infinite branch of the 7^{ic}. The other two segments, with their corresponding segments of v, generate two ovals external to one another, but situated in one of the seven regions formed by the intersections of g_7 and v. In this region must lie also the representative of the oval, if any, which is encircled by the generating oval. Hence the 7^{ic} of the first type must have three ovals lying in one of the 7 regions bounded by a segment of v and a segment of g_7, and the 7^{ic} of the second type (the one that has the pair of nested ovals) must have two ovals lying in one of these seven regions. In both cases the remaining six regions contain no ovals ; the arrangement of the other ovals is similar to the arrangement of those on the sextic from which they are derived. Hence *the* 8^{ic} *generated by the* 7^{ic} *of the first type will have one oval which encloses three others.* There is only one type of 8^{ic} obtained, but the generating oval may be that which encircles three others, or one which includes none. Thus, in passing to the 9^{ic}, there is a choice again between two modes of generation. *The* 8^{ic} *generated by the* 7^{ic} *of the second type has one oval enclosing two others and one including a single oval;* and two cases arise as before, according as the generating oval is the oval which encloses two others, or one which includes none. Hence, just as two types of the 7^{ic} were derived from the one form of the 6^{ic}, two types of the 9^{ic} are generated by each of the two forms of the 8^{ic}.

By exactly the same argument it can be shown that each type of the $2m - 2^{ic}$ gives rise to two types of the $2m - 1^{ic}$, determined by the character of the generating oval, g_{2m-2}, which may enclose a number of other ovals or may contain none at all. But since in the generation of the $2m^{ic}$ from the $2m - 1^{ic}$

the straight lines $l_1, l_2, \ldots . l_{2m-1}$ must cut the infinite segment of v, each type of the $2m - 1^{ic}$ can give rise to only one type of the $2m^{ic}$. In this way two types of the $2m^{ic}$ arise from each form of the $2m - 2^{ic}$ ($2m > 6$), and as the process is continued ($2m = 8, 10, \ldots . 2n$) there will arise 2^{n-3} types of the $2n^{ic}$. These, however, do not differ in the number of internal ovals. For with the exception of the ovals derived from v and the infinite branch of C_{2n-1}, the arrangement of the ovals of the $2n^{ic}$ is similar to that of the $2n - 1^{ic}$. The infinite branch of C_{2n-1} and v form $2n - 1$ regions, each bounded by a single segment of C_{2n-1} and a single segment of v, and in one of these regions must lie the representatives of all the ovals, if any, included by the generating oval of C_{2n-2}, as well as $\frac{1}{2}(2n - 2) - 1$ of the $2n - 3$ ovals which arise from segments of g_{2n-2} and v. No other of the $2n - 1$ regions contains an oval. Hence the oval on the $2n^{ic}$ derived from the segments of C_{2n-1} and of v which bound this region, will include $n - 2$ ovals and also those representing the ovals which were contained by g_{2n-2}. Thus whatever be the type of $2n^{ic}$, the number of its internal ovals exceeds by $n - 2$, the number on the $2n - 2^{ic}$.

For the 6^{ic} the number of internal ovals is 1,
for the 8^{ic}, $1 + 2$,
for the 10^{ic}, $1 + 2 + 3$,
for the 12^{ic}, $1 + 2 + 3 + 4$, etc.

Hence on *every* curve of even order ($2n$) with the maximum number of circuits there are $1 + 2 + 3 + 4 + \ldots . + n - 4 + n - 3 + n - 2$, i.e. $\frac{1}{2}(n-1)(n-2)$ internal ovals, and hence $n^2 + \frac{1}{2}(n - 1)(n - 2)$ external ovals.

Though the number of internal ovals is the same for all $2n^{ics}$ thus generated, the distribution differs from type to type. If on each $2m^{ic}$ ($2m = 2, 4, 6, \ldots . 2n - 2$) from which the $2n^{ic}$ is derived the generating oval be one which contains others, that is, if the first mode of generation be used throughout, all the internal ovals lie inside the same oval (Fig. $3_{(b), (d), (h)}$). If, however, the generating oval be one which encloses no other, then among the *additional* circuits formed in passing from a curve of order $2n - 2$ to a curve of order $2n$, there is just one oval which includes others, and this contains $n - 2$. Thus the $2n^{ic}$ which is derived by the second mode of generation throughout has its internal ovals distributed in $n - 2$ different ovals, in groups of $1, 2, 3, 4, \ldots . n - 3, n - 2$ (Fig. $3_{(c), (g), (o)}$). Combinations of the two modes of generation afford $(2^{n-3} - 2)$ other arrangements of the internal ovals, but there are certain restrictions to which the distribution is subject. No set of $1, 2, 3, \ldots ., n - 2$ ovals can be separated, and combinations can be made only of successive sets (Fig. $3_{(e), (f), (i), (j), (k), (l), (m), (n)}$).

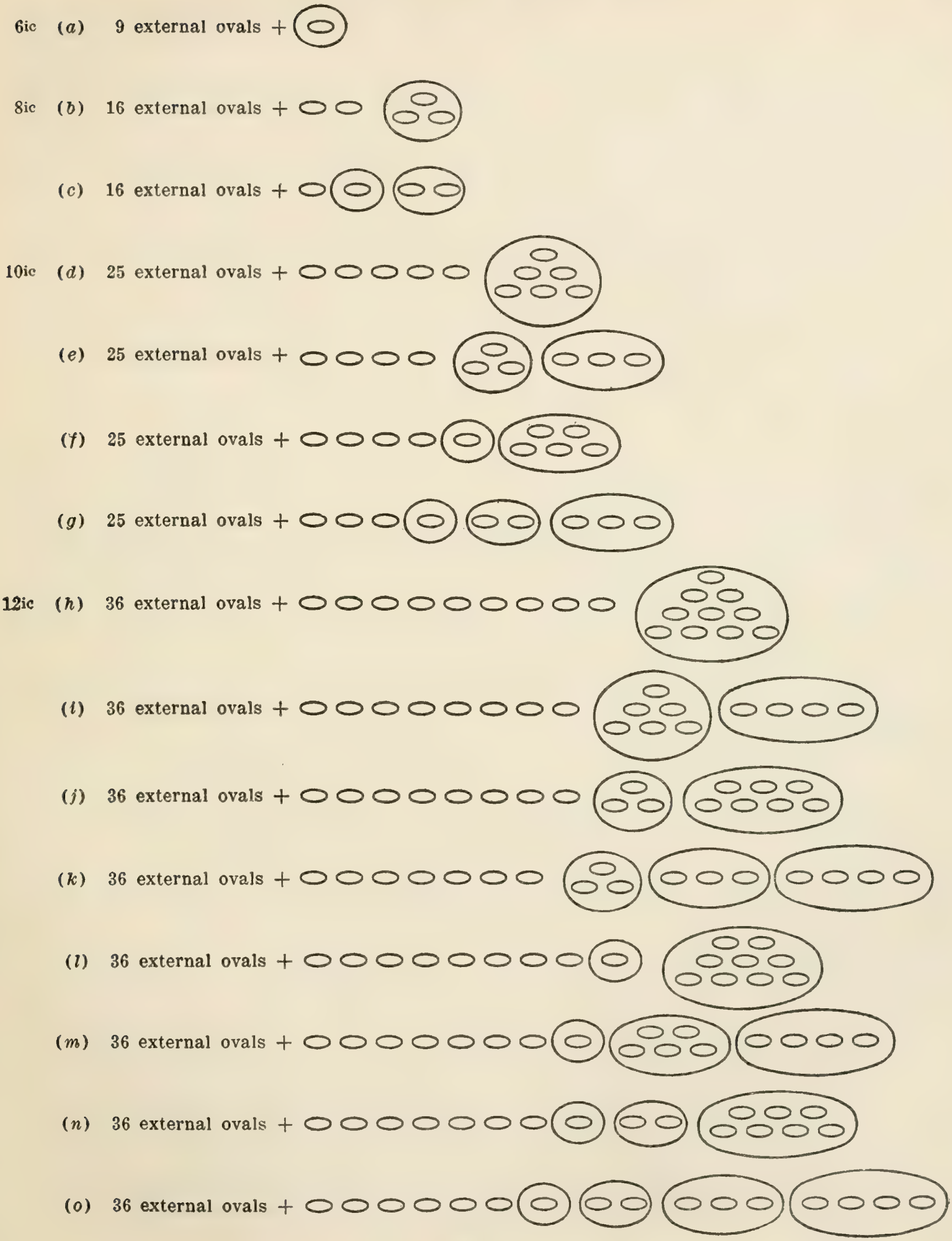
6ic (*a*) 9 external ovals +
8ic (*b*) 16 external ovals +
(*c*) 16 external ovals +
10ic (*d*) 25 external ovals +
(*e*) 25 external ovals +
(*f*) 25 external ovals +
(*g*) 25 external ovals +
12ic (*h*) 36 external ovals +
(*i*) 36 external ovals +
(*j*) 36 external ovals +
(*k*) 36 external ovals +
(*l*) 36 external ovals +
(*m*) 36 external ovals +
(*n*) 36 external ovals +
(*o*) 36 external ovals +

FIG. 3.

If the non-singular curve of even order has not the maximum number of circuits, the question arises, whether in this case the number of external ovals can exceed $n^2 + \frac{1}{2}(n-1)(n-2)$. The generating oval of C_{2n-2} with v gives rise to the infinite branch of C_{2n-1} and to $2n-3$ ovals, $n-2$ of which become internal ovals on the $2n^{ic}$. The remaining $n-1$ ovals, and all additional ovals arising from g_{2n-1} and v become external ovals. It is obvious that the $2n-2$ straight lines $l_1, l_2, l_3, \ldots l_{2n-2}$ could have been so chosen that the position of g_{2n-2} with regard to v would have been that indicated by Fig. 4, and hence that

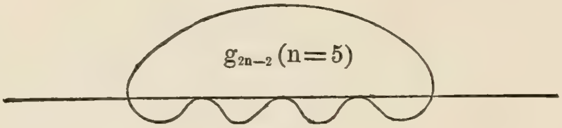

FIG. 4.

none of the $n-2$ internal ovals would have appeared on the $2n^{ic}$. This is the only way, however, in which the presence of internal ovals can be prevented without decreasing the number of external ovals, and this method admits of no increase in the number of the latter. Hence the conclusion can be drawn that no 6^{ic} derived by the Harnack process, can have more than 10 external ovals, no 8^{ic} more than 19 external ovals, no 10^{ic} more than $31, \ldots$, no $2n^{ic}$ more than $n^2 + \frac{1}{2}(n-1)(n-2)$, even though the number of circuits on the curve be less than the maximum number.

In support of the statement that on *curves of odd order*, the ovals may be so arranged that each lies outside the others, it was seen (p. 7) that from the 6^{ic} whose generating circuit included another oval, a 7^{ic} can be derived whose ovals lie external to one another. And in general from every $2n^{ic}$ whose generating oval includes *all* the internal ovals can be derived a $2n+1^{ic}$, all of whose ovals are external to one another. For as C_{2n+1} is generated from C_{2n}, the generating oval opens out, so to speak, to form with the infinite segment of v, the infinite branch of the $2n+1^{ic}$, thus leaving the ovals which it contained free of any encircling oval.

Curves of Even Order derived by the Hilbert Process.—For the types of $2n^{ics}$ which present themselves by the Harnack process, all the ovals which lie "inside" others satisfy the definition of "internal" ovals. Not so for curves with nested ovals; for within the annular regions bounded by two successive ovals of the nest, beginning with the outermost ring the expression, C_{2n}, is alter-

nately negative and positive. The internal ovals, then, are to be looked for only in these negative regions, that is, in the 1st, 3rd, 5th, ... etc., from the outside. The circuits lying in the 2nd, 4th, 6th, ... regions lie "inside" certain ovals but are themselves external ovals.

It has already been mentioned that the Hilbert method of generating curves with the maximum number of circuits, differs from the Harnack process only in the use of the ellipse instead of the straight line as auxiliary curve. The process, as given by Hilbert, applies to curves of both odd and even orders, but here only curves of even order will be considered. Let C_{2n} be a curve of even order with the maximum number of circuits, $p+1$, and the maximum number of nested ovals, $n-1$; and assume that an ellipse, E_2, can be drawn to enclose one or more of the nested ovals and cut one of the non-nested ovals, g_{2n}, in $4n$ points which have the same order of succession upon C_{2n} as upon the ellipse. On a segment, S, of E_2, but not that which with a segment of g_{2n} encloses the one or more nested ovals inside the ellipse, let $4n+4$ points be chosen and through these points let $2n+2$ straight lines, $l_1, l_2, \dots l_{2n+2}$, be drawn, connecting the first point with the second, the third with the fourth, and so on.* Then for a small value and the proper sign of δ the equation $C_{2n} \cdot E_2 + \delta \cdot \prod_{i=1}^{i=2n+2} l_i = 0$ represents a curve of order $2n+2$, which has the maximum number of branches, $p+1$, the maximum number of nested ovals, n, and satisfies all other conditions analogous to those assumed for the $2n^{ic}$.† Hence if a $2n^{ic}$ exists satisfying the assumed conditions, from it can be derived a $2n+2^{ic}$, subject to similar conditions, from this a $2n+4^{ic}$, and so on. For the case $2n=4$, such a curve does exist.

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* It can be seen, as in the preceding method, that the assumptions made for the $2n^{ic}$ are necessary for the maximum number of circuits, or for the maximum number of nested ovals on the $2n+2^{ic}$, or for the continuation of the process beyond the generation of C_{2n+2}. The assumption that the ellipse must enclose at least one of the nested ovals and the restriction made on the segment, S, are not given by Hilbert but are shown by Hulburt (*Topology of Algebraic Curves*, Bull. N. Y. Math. Soc. I, 1891-2, p. 197) to be necessary for the continuation of the process. Otherwise the curves of higher order would not have the maximum number of nested ovals. The restriction of the $4n+4$ points to the *same* segment is necessary in order that curves of order $>2n+2$ may have the maximum number of circuits.

† With the exception of the oval, g_{2n}, each circuit of C_{2n} gives rise to a circuit of C_{2n+2}. Also the boundaries of the $4n$ regions formed by the intersections of g_{2n} and E_2 generate $4n$ ovals. Hence *the total number of branches* $= \frac{1}{2}(2n-1)(2n-2)+4n = p+1$. The ovals arising from the nested ovals are nested, and one of the $4n$ ovals generated by the segments of g_{2n} and E is itself a nested oval. Hence the number of nested ovals $= \frac{1}{2}(2n-2)+1 = \frac{1}{2}(\overline{2n+2}-2)$. Moreover the ellipse encloses one or more of the nested ovals, and a non-nested oval, g_{2n+2}, cuts E_2 in $4n+8$ points whose order of succession is the same on the oval and the ellipse.

Let C_2 and E_2 represent two ellipses cutting each other in four real points. On a segment, S, of E_2 let 8 points be chosen and joined by the straight lines l_1, l_2, l_3, l_4, the 1st point from one end of the segment with the 2nd, the 3rd with the 4th, and so on. Then the equation $C_n \cdot E_2 + \delta \cdot \prod_{i=1}^{i=4} l_i = 0$, for a proper choice of δ, will represent a quartic satisfying the assumed conditions (Figs. 1, 2, Plate II). And therefore for all order values, $2n$, curves do exist satisfying similar conditions.

There is only one type of quartic obtained, but two cases arise from the two possible positions of the lines l_1, l_2, l_3, l_4 with reference to the auxiliary ellipse. If in the derivation of the 4^{ic} from the conic, C_2, no real points had been chosen on a segment, S, of E_2, the quartic would have consisted of two ovals inside E_2 and two ovals outside. Hence according as the 8 points chosen lie on a segment, S, outside C_2 or on a segment inside C_2, one of the two ovals inside E_2, or one of the two ovals outside E_2, becomes the generating oval. Each of these two quartics gives rise to a distinct type of 6^{ic} $(C_4 \cdot E_2 + \delta \cdot \prod_{i=1}^{i=6} l_i = 0)$ with the required properties (Figs. 3, 4, Plate II). It is easily seen that in the generation of each type of 6^{ic}, there are possible three essentially different positions of the 12 points which, taken in pairs, determine the 6 straight lines $l_1, l_2, \ldots l_6$. For one position, the ellipse, E_2, is cut by a non-nested oval of C_6 which would otherwise lie inside the ellipse; for another, by a non-nested oval which would otherwise lie outside the ellipse; for the third, by a nested oval. And, in general, the same possibilities arise in the derivation of the $2n^{ic}$ from the $2n - 2^{ic}$, thus affording three modes of generation of the $2n + 2^{ic}$ from a given $2n^{ic}$. The third case, however, as Hulburt points out, leads to curves with less than the maximum number of nested ovals. It will be found later that after an application of the third mode of generation, a fourth mode becomes possible. All four modes yield curves with the maximum number of circuits, but only the first and second admit also the maximum number of nested ovals.

If at each stage of the generation of the $2n^{ic}$ from the curves of lower order, one of the non-nested ovals inside the ellipse be taken as the generating oval, that is, if a *curve be derived* by *the first mode of generation* throughout, the circuits situated in the $n - 2$ annular regions determined by the nested ovals are distributed according to a perfectly regular scheme. The non-nested oval of the $2n - 4^{ic}$ which cuts the ellipse forms with the latter, $4n - 8$ regions in which

are generated $4n-8$ new ovals of the $2n-2^{ic}$. Of the $2n-4$ of these which lie inside the ellipse, one is a nested oval; another is to be taken as the generating oval; hence in the region bounded by the outer oval of the nest and those segments of the ellipse and the generating oval which give rise to the *new* nested oval of C_{2n}, there are situated exactly $2n-6$ ovals. Therefore the $2n^{ic}$ will have in the last annular region formed $2n-6$ ovals. Since one new annular region is formed at each stage of the generation, and the arrangement of the ovals lying in this region is not disturbed as curves of higher order are generated, there corresponds to each curve, C_{2m}, $(2m \overset{=}{>} 6)$ from which the $2n^{ic}$ is derived, one particular ring. For the 6^{ic}, the number of ovals in the ring between the two nested ovals $=0$; for the 8^{ic}, the number of ovals in the 1st, or innermost, and the 2nd rings $=0$, 2; for the 10^{ic} the number of ovals in the 1st, 2nd, and 3rd rings $=0$, 2, 4; and so on; for the $2n^{ic}$ the number of ovals in the 1st, 2nd, 3rd, ... $n-2^{th}$ rings $=0, 2, 4, 6, \ldots 2n-10, 2n-8, 2n-6$. But in the consideration of the number of internal or external ovals, that nested oval which forms the inner boundary of a ring itself, belongs to the group of ovals in that region; hence the foregoing scheme becomes

$$1, 3, 5, 7, \ldots\ldots\ldots 2n-9, 2n-7, 2n-5,$$

and these groups are alternately internal and external ovals, or vice versa. Therefore for $n-2$ even (Fig. 5, Plate II, n $=4$),

$$\text{the number of internal ovals} = 2n-5+2n-9+\ldots+7+3$$
$$= \tfrac{1}{2}(n-1)(n-2);$$

for $n-2$ odd,

$$\text{the number of internal ovals} = 2n-5+2n-9+\ldots+5+1$$
$$= \tfrac{1}{2}(n-1)(n-2).$$

If in the generation of the $2n^{ic}$ one of the non-nested ovals lying outside the ellipse be taken at each stage as the generating oval, that is, if the *curve be derived by the 2nd mode of generation throughout,* there is obtained a similar arrangement of circuits in the annular regions. Beginning with 3, however, the series is reversed ($1, 2n-5, 2n-7, 2n-9, \ldots 7, 5, 3$), since by this process after the generation of the 6^{ic}, the nest is built up from the outside inward; and the innermost ring contains not only one oval in accordance with the scheme, but also all, save one, which by the other process lay in that part of the plane exterior to the nest. Therefore for $n-2$ *even* when within the innermost ring,

the expression, C_{2n}, is positive (Fig. 6, Plate II), the number of internal ovals $= \frac{1}{2}(n-1)(n-2)$, but for $n-2$ *odd* the number of internal ovals

$$= n^2 + \tfrac{1}{2}(n-1)(n-2) - 1 > \tfrac{1}{2}(n-1)(n-2).$$

Combinations of the 1st and 2nd modes of generations give other types of curves. At each stage of the development of the curve, C_{2n}, from the conic, C_2, either mode of derivation may be adopted; hence from the quartic of which there is only one type, are derived two 6^{ics}, from each 6^{ic}, two 8^{ics}, and so on; the number increasing in geometrical ratio as n increases. Therefore for the $2n^{ic}$, if the two regular types just discussed be included, the number of types of curves with the maximum number of circuits and the maximum number of nested ovals, is 2^{n-2}. By a combination of the two modes, the nest is built up alternately from the inside outward and from the outside inward. If the curve, C_{2m-2}, is derived throughout the process by the first mode of generation, all of its nested ovals lie inside the ellipse, E_2. For the generation of the curve, C_{2m}, from this, let a change be made to the 2nd mode; then the *new* nested oval of C_{2m} lies outside the ellipse, and the annular region which corresponds to the curve, C_{2m}, is the one in which the ellipse is situated. As curves of higher order are generated all the *new* ovals appear in this region, and from it are cut off successively the *new* annular regions lying inside or outside the ellipse according as they are formed by the 1st or 2nd mode of generation. The curve is built up in such a manner that there are formed sets of annular regions, (1) a inside the ellipse, (2) a' outside the ellipse, (3) b inside the ellipse, (4) b' outside the ellipse, and so on. Except in those rings which are formed at stages where a change in the process occurs, the number of ovals in each annular region is the same as that in the corresponding region when one type of generation is used throughout,—that is in the last annular region formed whether outside or inside the ellipse there are $2n-6$ ovals. But consider the ring formed in the derivation of C_{2m} from C_{2m-2} where a change is made from the 1st to the 2nd mode of generation. The outer nested oval of C_{2m-2} is one of the $2m-4$ ovals arising from segments of g_{2m-4} and E_2, and lying inside the ellipse; and the generating oval of C_{2m-2} is one of the $2m-4$ ovals outside the ellipse. Thus $2m-5$ ovals are left in the region between the ellipse and the outer nested oval of C_{2m-2}. Hence as C_{2m} is derived from this $2m-2^{ic}$, an annular region is formed which contains $2m-5+2m-2$ ovals inside the ellipse and $2m-3$ lying outside. If the second process is continued through

the generation of C_{2p-2}, the number of ovals in each new annular region formed follows the regular scheme up to this stage, and the appearance of every group of $2q - 6$ ovals in a ring outside the ellipse ($2q = 2m + 2, 2m + 4, \ldots 2p - 2$), is accompanied by the appearance of a group of $2q - 4$ ovals in the region containing E_2 and inside the ellipse. Hence this ring contains

$2m - 5 + \sum_{m+1}^{p-1}{}_q\, 2q - 4 + 2p - 4$ ovals inside the ellipse and $2p - 5$ outside.

If at this stage a change is made back to the first type of generation, one of the $2p - 4$ ovals inside the ellipse must be taken as the generating oval, g_{2p-2}, and hence in the new annular region formed for C_{2p}, a region which lies inside the ellipse, there are $2m - 5 + \sum_{m+1}^{p-1}{}_q\, 2q - 4 + 2p - 5$ ovals, and in the region which contains the ellipse $2p - 3$ ovals inside the ellipse and $2p - 2 + 2p - 5$ outside. Hence as C_{2n} is derived, if there is a change from the 1st mode of generation to the 2nd in the derivation of C_{2m}, from the 2nd to the 1st in the derivation of C_{2p}, from the 1st to the 2nd in the derivation of C_{2s}, and so on, the scheme of arrangement of the ovals in the annular regions beginning with the innermost ring is represented by the following sets of groups, $a, a', b, b', c, c', \ldots,$

$$(a)\,|\,0, 2, 4, 6, 8, \ldots 2m - 8\,|,\ (b)\,|\,(2m - 5 + \sum_{m+1}^{p-1}{}_q\, 2q - 4 + 2p - 5),$$
$$2p - 4,\ 2p - 2, \ldots 2s - 8\,|,$$

$$(c)\,|\,(2s - 5 + \sum_{s+1}^{t-1}{}_q\, 2q - 4 + 2t - 5),\ 2t - 4,\ 2t - 2, \ldots 2n - 6,$$
$$(2n - 2 + 2n - 3 + \sum_{t+1}^{n}{}_q\, 2q - 4 + 2t - 5)\,|,$$

$$(b')\,|\,2t - 8, \ldots 2s - 2,\ 2s - 4,\ (2s - 5 + \sum_{p+1}^{s-1}{}_q\, 2q - 4 + 2p - 5)\,|,$$

$$(a')\,|\,2p - 8, \ldots 2m - 2,\ 2m - 4\,|;$$

or if in the enumeration of the ovals in the annular regions the nested oval

which forms the inner boundary of the ring be included, the foregoing scheme becomes

$$(a)|1, 3, 5, 7, \ldots 2m-7|,\ (b)|(2m-5+\sum_{m+1}^{p-1}{}_q\, 2q-4+2p-4),$$
$$2p-3,\ 2p-1, \ldots 2s-7|,$$

$$(c)|\ (2s-5+\sum_{s+1}^{t-1}{}_q\, 2q-4+2t-4),\ 2t-3,\ 2t-1, \ldots 2n-5,$$
$$(2n-1+2n-3+\sum_{t+1}^{n}{}_q\, 2q-4+2t-5)|,$$

$$(b')|\ 2t-7, \ldots 2s-1,\ 2s-3,\ (2s-4+\sum_{p+1}^{s-1}{}_q\, 2q-4+2p-5)|,$$
$$(a')|\ 2p-7, \ldots 2m-1,\ 2m-3|.$$

If the 2nd mode of generation is applied first, then the region which includes the ellipse corresponds to the 6^{ic}. The foregoing scheme applies, but $2m$ has the value 6, and group (a) disappears.

If the annular regions be considered in the order of their formation but that region in which the ellipse is situated be left to the last, the series of numbers representing the groups of ovals is $(a)|1, 3, 5, 7, \ldots 2m-7|$, $(a')|2m-3,\ 2m-1 \ldots 2p-7|$, etc. Comparison with the regular series obtained when the first mode of generation is employed throughout shows that after $2m-7$, each number in this series is greater by 2 or by a quantity exceeding 2, than the corresponding number in the regular series. If in the last region of each set, $a, a', b, b', \ldots$ the expression, C_{2n}, is negative, or more briefly if the last region in each set is negative except in that set in which the ellipse is situated, the series which represents the number of *internal* ovals is

$$\underset{\text{or } 3,\,7,\,11,}{1,\ 5,\ 9,} \ldots 2m-11,\ 2m-7,\ 2m-3,\ 2m+1, \ldots 2p-7,\ 2p-3, \ldots\ldots$$
$$2s-7,\ 2s-3, \ldots\ldots$$

in exact agreement with the regular series representing the number of internal ovals when the 1st mode of generation is used throughout. For though the first region of (a') is negative the number of its ovals exceeds by 4 the number in the last region of (a), and as the first regions of all the other sets $b, b', c, c', \ldots$ are necessarily positive there can be no break in the series as passage is made from set to set. But when there is a change made from one mode of generation to the

other, two regions which are formed consecutively are separated by other regions as the process is continued, and hence the expression C_{2n} may be negative in both these regions. This apparently introduces the possibility of the existence of a series in which some of the numbers are less than the corresponding numbers in the regular series, and in this case the number of internal ovals might be less than $\frac{1}{2}(n - 1)(n - 2)$. For example, if the last region of (b') and the first region of (c) are both negative, the series will contain a sequence of the form, $2t - 7$, $(2t - 4 + \ldots)$ $2t - 1$, $2t + 3, \ldots,$ (A), in which the numbers beginning with $2t - 1$ are less by 2 than the corresponding numbers in the sequence $2t - 7$, $2t - 3$, $2t + 1$, $2t + 5 \ldots\ldots$ But the conditions for the existence of such a sequence as (A) demand the existence of a preceding sequence of the form $2l - 9$, $2l - 3$, $2l + 1$, $2l + 5$ instead of the regular sequence $2l - 9$, $2l - 5$, $2l - 1$, $2l + 3$. Thus the numbers in the series which precede such a sequence as (A) exceed by 2 the corresponding numbers in the regular series, and hence the numbers following the sequence will be the same as the corresponding numbers in the regular series.

Suppose such a sequence as $2p - 7$, $(2p - 4 + \ldots)$, $2p - 1$, $2p + 3$ occurs in the passage from the set (a') to the set (b). The last region of (a') and the first region of (b) are both negative. Since the first region of (b) is negative, the last region of (a) must be positive. The last negative region of (a) then contains $2m - 9$ ovals, and hence in the series of numbers, representing the groups of internal ovals there is a spring from $2m - 9$ to $2m - 3$, preceding the succession $2p - 7$, $(2p - 4 + \ldots)$, $2p - 1$, $2p + 3$.

$(a) \ldots\ldots\ldots + (b) - \ldots\ldots\ldots - (c) \ldots\ldots\ldots + (d) \ldots\ldots\ldots (e) \ldots\ldots\ldots$

$(d') \ldots\ldots\ldots (c') + \ldots\ldots\ldots + (b') - \ldots\ldots\ldots (a') - \ldots\ldots\ldots -$

No such break can occur again in the series so long as the last region in each set considered is negative, for as a prerequisite to such a sequence as the above, the first region of some set must be negative and this is impossible unless the last region in the preceding contiguous set be positive. Let the first set whose last region is positive be (c). The last region of (b') is negative, hence the first region of (c') is positive. Therefore in the passage from (c) to (c') there is a jump over two annular regions, that is, from a group of $2l - 9$ internal ovals to a group of $2l - 3$. If the last region of (c') is positive, the series representing the groups of internal ovals goes on regularly except that in the 1st region of

(d), which is negative since the last region of (c) is positive, the number of internal ovals is greater than it would have been had there been no change in the mode of generation. This regularity continues until the last region in some set is negative, and at this stage occurs again a sequence of the form $2r - 7$, $(2r - 4 + \ldots)$, $2r - 1$, $2r + 3 \ldots$. In every case a sequence of the above form is preceded by a succession of the form $2l - 9$, $2l - 3$, $2l + 1$, $2l + 5$. Thus in the series representing the groups of internal ovals there may be numbers exceeding but none less than the corresponding numbers in the regular series. Hence since the series of numbers which represent the groups of internal ovals on curves derived by both modes of generation is either greater than or equal to the series obtained when the 1st mode of generation is used throughout, the number of internal ovals $\overline{\gtrless} \frac{1}{2}(n - 1)(n - 2)$.

The *third mode of generation* in which the new nested oval is taken as the generating oval, is not applicable to the generation of curves of degree lower than 8, for the 6^{ic} is the first curve whose generating oval can be a nested oval. Hence the first application of the process must be preceded by the use of the first or second mode of generation. Each application of the process reduces the number of nested ovals by 2. For suppose the new nested oval of C_{2m-2} cuts the ellipse; then on C_{2m} there is no *nested* oval representing this, and moreover none of the $4m - 4$ ovals arising from the segments of the ellipse and this generating oval encloses either the ellipse or the nested ovals lying inside the ellipse; that is, the nested oval contributed to the nest by C_{2m-2} disappears as such, and no new nested oval is added by C_{2m}. It is evident that since there is no nested oval among the new ovals formed at this stage, the third mode of generation cannot be applied twice in succession. Thus the different types of curves which are derived in part by the 3rd mode of generation are those obtained by the 1st and 3rd modes of generation or by the 2nd and 3rd, or by combinations of all three modes, or by combinations of these with the fourth. If the curves be derived by the 1st and 3rd *modes of generation*, let the 3rd mode be introduced for the first time for the generation of the curve C_{2m}. The new nested oval of C_{2m-2} lies inside the ellipse, and in the new annular region formed there are $2m - 8$ ovals, and in the region between the nested ovals and the ellipse there are $2m - 5$ ovals. The new nested oval of C_{2m-2} is to be taken as the generating oval. Hence in one of the $4m - 4$ regions bound by segments of E_2 and g_{2m-2} there are $2m - 5$ ovals, and therefore of the $4m - 4$ new ovals appearing on C_{2m}, one contains $2m - 5$ others, and this lies inside the ellipse. The

remaining $4m-5$ are distributed with respect to the ellipse just as the corresponding $4m-5$ ovals would have been distributed had the first mode of generation been used instead of the third; so the process can be continued as if the third type had not been introduced except that a nested oval cannot be taken as the generating oval. Hence the general arrangement of the ovals of the curve, C_{2m}, which is derived throughout by the 1st method of generation is affected by the introduction of the third method for the generation of C_{2m} only in the following manner. *The ring which by the continuation of the first process would have contained* $2m-6$ *ovals disappears, and in place of the two nested ovals bounding it,*

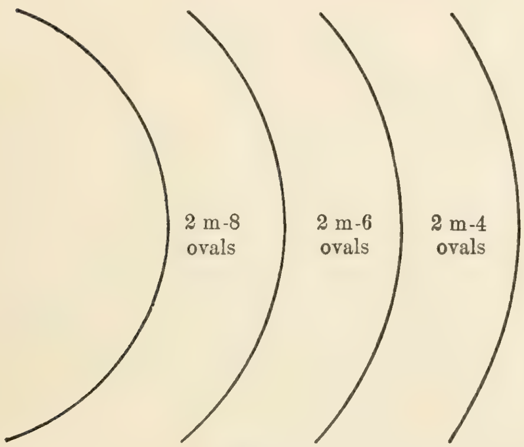

Fig. 5.

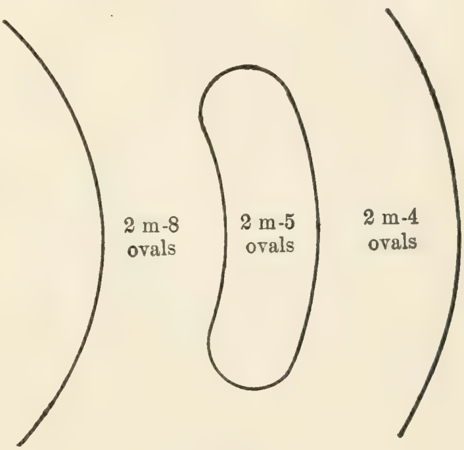

Fig. 6.

with the $2m-6$ *ovals lying between them, there is now one oval enclosing* $2m-5$ *others.* Thus three annular regions are thrown into one, and the arrangement of circuits represented by Fig. 5 becomes that indicated by Fig. 6. Or if the process stops here, the two outer rings become one with the region outside the nest.

If the annular region which is composed of the three is positive, it contributes the same number of internal ovals as the middle region would have contributed, if the three were distinct, namely $2m-5$. If negative, it yields the same number that would have been given by the other two regions. In the enumeration of the internal ovals the nested oval which forms the inner boundary of a negative ring must be included, and though the two regions in question would yield two such internal ovals and the composite region only one,

yet as compensation for the other there is the oval which contains the $2m - 5$ other ovals. If the new nested oval of C_{2m+2} be taken as the generating oval for the derivation of C_{2m+4}, that is, the 3rd mode of generation applied again, then there are combined into one the five annular regions which would have appeared had the first mode of generation been used throughout; but in this composite region there are besides $2m - 8 + 2m - 4 + 2m$ ovals, one oval enclosing $2m - 5$ others, and another oval encircling $2m - 1$ ovals. The arrangement is indicated by Fig. 7. It is evident that the number of internal ovals is not altered by the introduction of the third mode of generation, and hence every

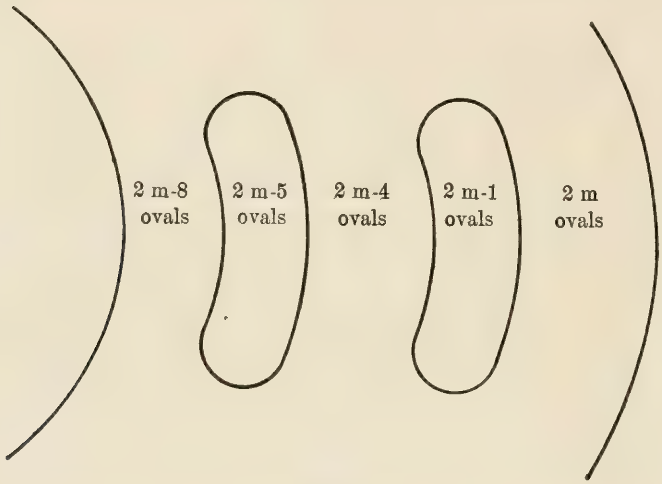

FIG. 7.

combination of the 1st and 3rd modes of derivation gives a curve with $\frac{1}{2}(n - 1)(n - 2)$ internal ovals.

The arrangement of the circuits of a curve derived by the 2nd mode of generation or by a combination of the 1st and 2nd modes of generation is modified by the introduction of the 3rd mode in a manner similar to that in the preceding case. Suppose the 3rd mode is introduced for the derivation of C_{2m}. Either the 1st or 2nd process must have been used for the derivation of C_{2m-2}. If this process had been continued for C_{2m} also, an annular region would have been formed containing $2m - 6$ ovals between the two nested ovals. The use of the 3rd method causes the disappearance of these two *nested* ovals as such and introduced in the place of them, with the $2m - 6$ ovals between them, one oval enclosing $2m - 5$ others. And this is the only alteration produced. The

conclusion is easily deduced as in the preceding case that the number of internal ovals on curves derived by the 2nd and 3rd modes of generation or by all three modes combined, is not less than $\frac{1}{2}(n-1)(n-2)$.

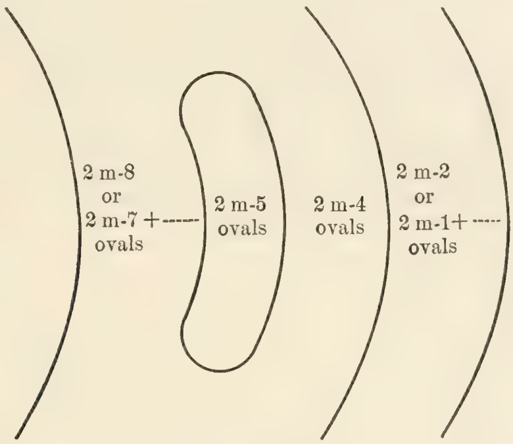

FIG. 8.

Although the 3rd mode of generation cannot be applied twice in succession, it need not be followed by the first or second, for the ellipse may be cut by the oval which contains $2m-5$ others. This introduces a *fourth mode of derivation,*

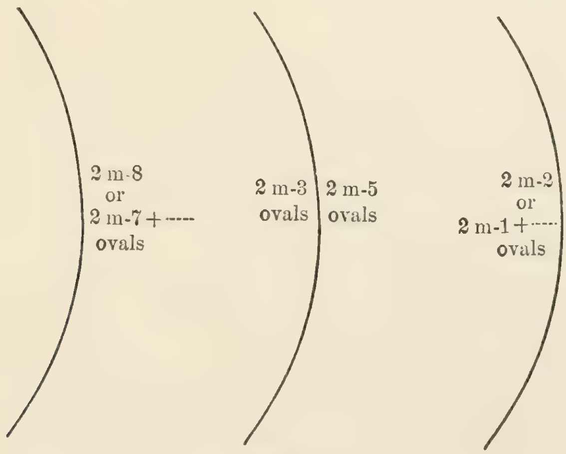

FIG. 9.

closely related however to the 3rd type, for though the generating oval is not one of the nested ovals, yet it does include others. It is not difficult to see that if the arrangement of the ovals in the annular regions formed at the stages

C_{2m-2}, C_{2m}, C_{2m+2}, C_{2m+4} on a curve derived by the 1st, 2nd and 3rd modes be represented by Fig. 8, the use of the 4th mode for the derivation of C_{2m+2} will modify the arrangement to that indicated by Fig. 9. That is, there is a combination in pairs of the four annular regions which would have appeared if instead of the 3rd and 4th modes of generation the one preceding the 3rd had been used; the first and third are united, and the second and fourth. If one region is negative, the other is positive, so that the same number of internal ovals is obtained as if the ovals were distributed in the four regions. Hence in this case, the introduction of the 4th mode of generation does not decrease the number of

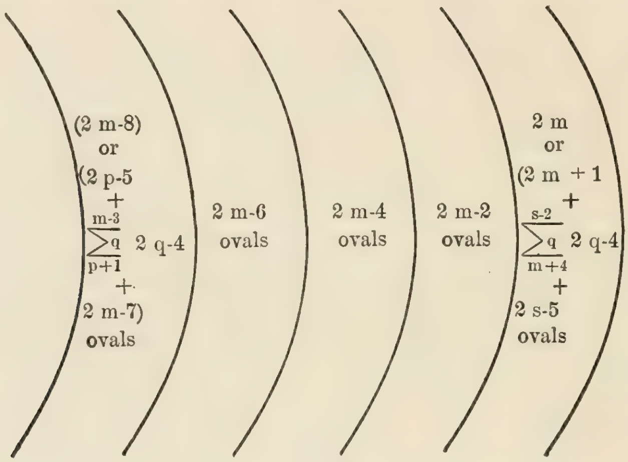

FIG. 10.

external ovals. If the 4th mode of generation be followed by the 3rd instead of the 1st or 2nd, then a somewhat different arrangement is produced. There are combined into one the five annular regions which would have appeared, if the mode by which C_{2m-2} was derived had been continued for the generation of C_{2m}, C_{2m+2}, C_{2m+4}, C_{2m+6}. Instead of the arrangement represented by Fig. 10, the result would be that indicated by Fig. 11.

If all possible combinations be made of the four modes of generation, various arrangements of the circuits are obtained, but the investigation of the preceding combinations makes it evident that the number of internal ovals on the derived curve is equal to the number on a curve which is generated by the 1st and 2nd modes only and hence is not less than $\frac{1}{2}(n-1)(n-2)$.

The appearance of a curve derived by the 1st or 2nd mode of generation, or by a combination of the two, is only slightly modified by an occasional intro-

duction of the 3rd mode of derivation; but an extensive use of the 3rd method gives a curve differing greatly in form from that derived by the 1st or 2nd mode alone, or by the 1st and 2nd together. For example, if for the derivation of the 8^{ic} from the 6^{ic} which is generated by the 1st mode of generation, the 3rd mode be employed, and for the derivation of curves of higher order the 4th and 3rd modes be used alternately, then no nest whatever is built up. The type of curves obtained is the same as that derived by the exclusive use of the 1st mode of generation in the Harnack process. The 8^{ic} has one oval containing three, and the remaining ovals are external to one another; the 10^{ic} has one oval

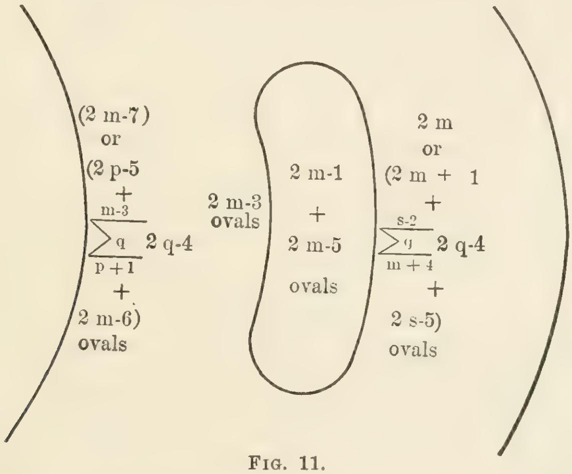

FIG. 11.

containing 6 and the remainder external to one another, and so on. If the 1st mode of generation is combined with the 3rd and 4th, but after the generation of the 8^{ic} not applied twice in succession, there are derived in this way $m-2$ other types of curves of order $4m$, and $l-2$ other types of curves of order $2(2l-1)$, which agree with types obtained by the Harnack process. But no other types of the Harnack curves are derived by the use of the 2nd mode instead of the 1st or by the use of both the 1st and 2nd.

It is evident from the method of generation that the number of internal ovals can be diminished in the same manner as in the generation of curves by the Harnack process, and also that this diminution can in no case be accompanied by an increase in the number of external ovals. Therefore the number of external ovals is not greater than $n^2 + \frac{1}{2}(n-1)(n-2)$ even if the number of circuits falls short of the maximum number.

Curves Derived by the Modified Forms of the Hilbert Method.—The use of the *hyperbola* or *parabola* as auxiliary curve, since these can be projected into the ellipse, can certainly yield no type of $2n^{ic}$ differing from those obtained by the Hilbert process. Neither does the *degenerate conic.* This, however, requires a special proof. But when the pair of straight lines is substituted for the ellipse as the auxiliary curve, passage can be made at any time to the Harnack process, by the mere disregard of one of the lines, and a return to the original mode of derivation can be effected after the generation of any curve of even order. Thus new types of curves may arise by a combination of the two processes, but it can certainly be shown that no type of $2n^{ic}$ obtained has less than $\frac{1}{2}(n-1)(n-2)$ internal ovals.

If it could be shown that all non-singular curves with the maximum number of circuits can be generated by this method of small variation, the proof of the validity of the law for these remaining cases would establish it for all non-singular curves of even order. But as yet there is no formal proof that the list of such curves is exhausted by the types considered.

CONCLUSION.

There are several other forms in which the theorem can be stated that are of interest, either as facts resulting from the theorem if established in its preceding form, or as statements which may afford a better starting point for the proof of the theorem. A few of these equivalent forms are obtained by a consideration of the *Theory of the Characteristic*, which though apparently yielding no results toward the proof of the theorem, bears a most interesting relation to the problem. The theory as given by Kronecker* is purely algebraic; he proved that for any system of algebraic functions satisfying certain conditions, there exists a number derived algebraically which is invariant for that system. Dyck,† however, was led by a study of Kronecker's investigations to a *geometrical* definition of a characteristic number associated with a manifold, a number which is built up as the manifold itself is developed. He showed that if the manifold can be expressed algebraically, it can be developed by processes which also are capable of algebraic expression. The whole geometrical configuration thus

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* *Ueber Systeme von Functionen mehrerer Variabeln*, Werke I, 1869, pp. 175–212, 213–226. *Ueber die Characteristik von Functionen Systemen*, Werke II, 1878, pp. 71–82.

† *Beiträge zur Analysis Situs*, I, II, III, Berichte der K. Sächs. Gesellschaft der Wissenschaften (Math. Phys. Classe) 1885, 1886, 1887. Math. Ann., vol. 32.

introduces a system of algebraic functions subject to certain conditious. Dyck proved that the characteristic number associated with the manifold can be derived from this system of functions and that the number is identical with the Kronecker characteristic of the system.

In Dyck's Theory the manifold is regarded as made up of elements, and to each element is assigned the characteristic $+1$. Whatever be the process of generation of the manifold,

(*a*) the appearance of a new element contributes $+1$ to the characteristic;

(*b*) the vanishing of an element contributes -1;

(*c*) the separation of an element into two pieces contributes $+1$;

(*d*) the joining of two elements, or of two parts of the same element, contributes -1.

For a one-dimensional manifold,—that is, a figure composed of lines,—the element is a broken piece of a curve. For a two-dimensional manifold, the element may be a part of a plane or of a surface that can be developed as a plane. The manifold suggested by any problem relating to the arrangement of the circuits on a plane curve, $f(xy) = 0$, of even order, with no singularities and with the maximum number of circuits, is obviously the two-dimensional manifold determined as the parts of the plane lying inside the curve,—that is, the parts of the plane where $f < 0$. The element is a piece of the plane bounded by a non-singular closed circuit (Fig. 12), and to this is assigned the characteristic $+1$. If the figure is initially non-existent, its characteristic is zero, and as the manifold is generated the characteristic is increased by unity as an element appears or separates into two, and is diminished by unity as an element vanishes, or as two elements or two parts of the same element unite. Therefore the characteristic of the manifold is equal to the sum of the characteristics of the separate parts which make up the manifold, and is independent of the mode of generation. Thus in the given manifold a part of the plane bounded by a single oval has the characteristic $+1$, whether it arises from a single element in the process of generation or from the union of several elements. Hence each external oval on the curve contributes to the manifold a region whose characteristic is $+1$. If two parts of such a piece of the plane unite and thus enclose in the midst of a region in which f is negative, a region where f is positive, their union is

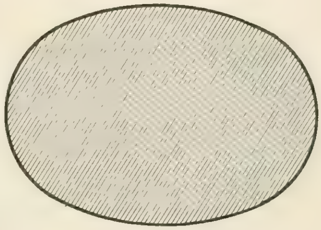


FIG. 12.

marked by -1 in the characteristic. But this is just the way in which an external oval makes its appearance. Hence the presence of each internal oval on the curve diminishes the characteristic of that piece of the plane to which it belongs by unity. The whole number of circuits on the $2n^{ic}$ is $n^2 + \frac{1}{2}(n-1)(n-2)$, and hence according to the theorem which ascribes the minimum limit $\frac{1}{2}(n-1)(n-2)$ to the number of internal ovals, the characteristic cannot exceed $n^2 + \frac{1}{2}(n-1)(n-2) - \frac{1}{2}(n-1)(n-2)$ or n^2, if the maximum number of circuits is present. It has been noted that a decrease in the number of internal ovals cannot be accompanied by an increase in the number of external ovals. Hence the characteristic cannot be greater than $n^2 + \frac{1}{2}(n-1)(n-2)$ even if the number of branches is not the maximum number.

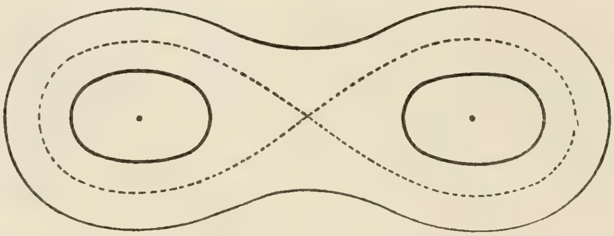


FIG. 13.

The process of generation of such a manifold can be expressed analytically by considering the manifold as one of a singly infinite system of manifolds. Let the curve $f = 0$ be obtained as one of the pencil $f = \lambda$. Then the region where f is negative decreases as λ decreases, and for some value of λ sufficiently near $-\infty$, the curve disappears altogether. The characteristic of the corresponding manifold is zero. Let λ increase from this value. The curve makes its appearance as an isolated point spreading into a circuit as λ continues to increase. Other circuits also may come into existence in the same way. Thus an isolated point gives rise to a part of the plane and hence has the value $+1$ for the characteristic. Two circuits may unite, producing a node (Fig. 13) and thus join two pieces of the plane together, or a single circuit may cut itself (Fig. 14), and so unite parts of the same region of the plane. In either case the node is marked by -1 in the characteristic. As λ increases, an internal oval shrinks until it becomes an isolated point and then passes out of existence, or it may cut itself in such a way that at the next stage it breaks into two internal ovals. In all cases the isolated points which present themselves in the region $f < 0$ contribute $+1$ to the characteristic, and the nodes -1. The singular points of the system

are given by $f_1 = 0, f_2 = 0$, and are nodes with real tangents or isolated points according as $\begin{vmatrix} f_{11} & f_{12} \\ f_{12} & f_{22} \end{vmatrix}$ is negative or positive. If the notation $\left[\begin{vmatrix} f_{11} & f_{12} \\ f_{12} & f_{22} \end{vmatrix}\right]$ be used to represent $+1, 0, -1$ according as the determinant is positive, zero, or negative, the characteristic of the manifold can be written $X = \Sigma \left[\begin{vmatrix} f_{11} & f_{12} \\ f_{12} & f_{22} \end{vmatrix}\right]$, the summation taken over all points $f_1 = 0, f_2 = 0, f < 0$; and Dyck shows that this is the same as the Kronecker characteristic for the system of functions f, f_1, f_2. Since the characteristic cannot exceed n^2 if $f = 0$ be a curve (order $2n$) with the maximum number of circuits, it follows that the number of isolated points passed over in the system $f = \lambda$ from $\lambda = -\infty$ to $\lambda = 0$ exceeds the

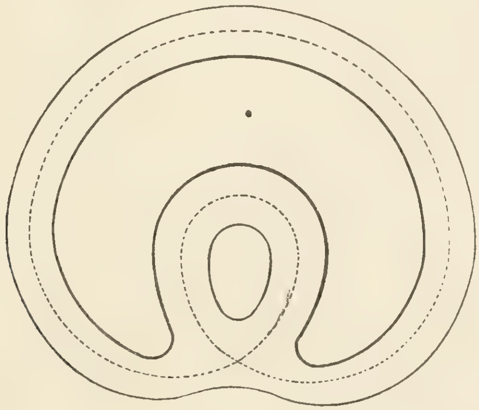


FIG. 14.

number of nodes by a quantity less than, or equal to n^2. And if $f = 0$ has not the maximum number of circuits, the excess of the number of isolated points over the number of nodes in the region $f < 0$ is either less than or equal to $n^2 + \frac{1}{2}(n-1)(n-2)$.

The relations between the critic centres of the pencil $f = \lambda$ thus obtained in applying the Theory of the Characteristic to an interpretation of the problem are interesting, but afford no clue to the solution. There is even some indication that the theory is not the most promising instrument of proof, for this is applicable to curves of even order only, and though the theorem on the minimum limit of the number of internal ovals is stated for these curves alone, it most probably can be extended to include curves of odd order also. It has been seen that on a non-singular $2n + 1^{ic}$ with the maximum number of branches the ovals may lie each outside the others; but even in this case they may satisfy

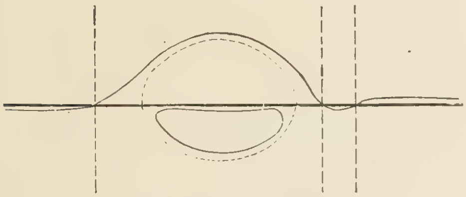
the definition of "internal ovals," for the odd circuit divides the plane into two regions both infinite, in one of which C_{2n+1} is positive, and in the other, negative. And as a matter of fact, no $2n+1^{ic}$ with fewer than $\frac{1}{2}(n-1)(n-2)$ internal ovals presents itself directly by either of the two processes of generation discussed.

There are still other forms in which the theorem can be stated, the most interesting of which is perhaps one relating to the number of regions into which the plane is divided by the curve. It has been shown that if the maximum number of branches is present, then the curve must have at least $\frac{1}{2}(n-1)(n-2)$ *internal* ovals, and whether the number of branches present is the maximum or less than the maximum, the curve can not have more than $n^2 + \frac{1}{2}(n-1)(n-2)$ *external* ovals. By a similar line of reasoning it can be proved that whatever the number of circuits, the number of internal ovals can not exceed $n^2 + \frac{1}{2}(n-1)(n-2) - 1$; and hence if the maximum number of circuits is present the number of external ovals can not fall below $\frac{1}{2}(n-1)(n-2) + 1$.

Therefore if the maximum number of branches is present,
the number of regions in which the expression C_{2n} is positive $\overline{>}\ \frac{1}{2}(n-1)(n-2)+1$,
and the number of regions in which C_{2n} is negative $\overline{>}\ \frac{1}{2}(n-1)(n-2)+1$;
and whatever the number of circuits,
the number of regions in which C_{2n} is positive $\overline{<}\ n^2 + \frac{1}{2}(n-1)(n-2)$
and the number of regions in which C_{2n} is negative $\overline{<}\ n^2 + \frac{1}{2}(n-1)(n-2)$.
From these statements, it seems that any limitation on the arrangement of the circuits is of a dual nature; and it is worthy of note that no modification of these statements is necessary, if the sign of C_{2n} be so chosen that it is negative at infinity.

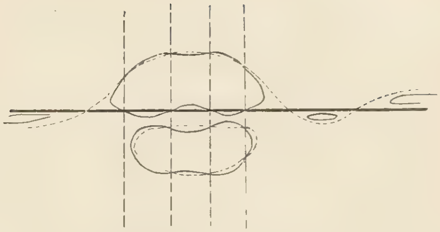
The statement of the theorem in this last form suggests that there may be some underlying relation to the theory of Multiply-connected Surfaces.

In the figures of Plates I and II the curves are much distorted, inflexions being inserted where none exist, in order to bring the figures within the scale of the paper. The figures represent the distribution of the ovals of the curves accurately only with respect to the *number* in different regions of the plane. In the drawing of the figures representing curves of the type, $C_{2n} \equiv C_{2n-2} \cdot E_2 + \delta \prod_{i=1}^{i=2n} l_i = 0$, the straight lines l_i are omitted.



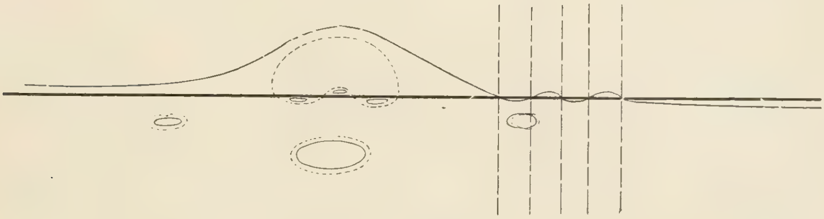
$$\underline{C_3} \equiv C_2 \cdot \underline{v} + \delta \prod_{i=1}^{i=3} \underline{l_i} = 0.$$

FIG. 1.



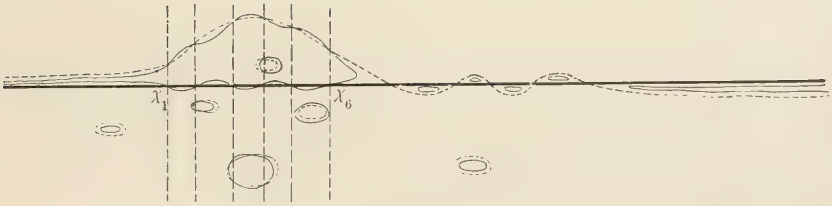
$$\underline{C_4} \equiv C_3 \cdot \underline{v} + \delta \prod_{i=1}^{i=4} \underline{l_i} = 0.$$

FIG. 2.



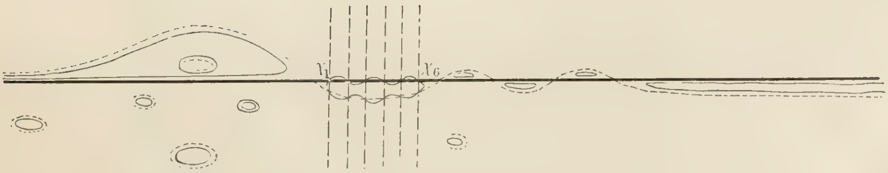
$$\underline{C_6} \equiv C_4 \cdot \underline{v} + \delta \prod_{i=1}^{i=5} \underline{l_i} = 0.$$

FIG. 3.



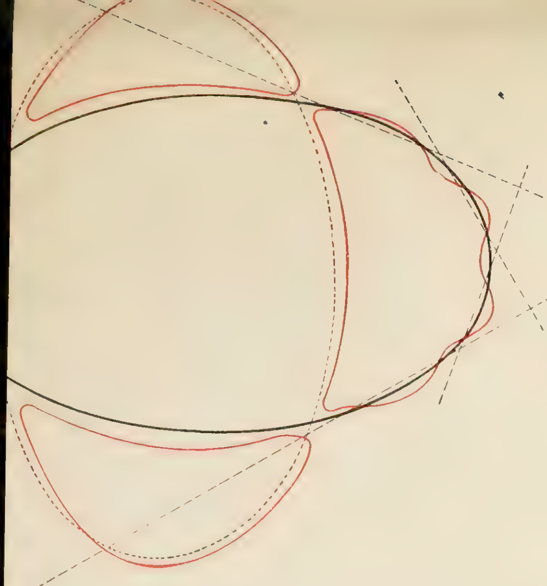
$$\underline{C_6} \equiv C_5 \cdot \underline{v} + \delta \prod_{i=1}^{i=5} \underline{l_i} = 0.$$

FIG. 4.



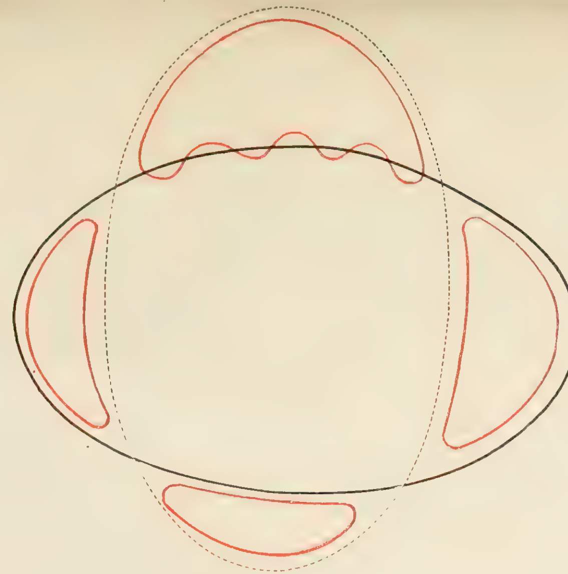
$$\underline{C_6} \equiv C_5 \cdot \underline{v} + \delta \prod_{i=1}^{i=6} \underline{l_i} = 0.$$

FIG. 5.



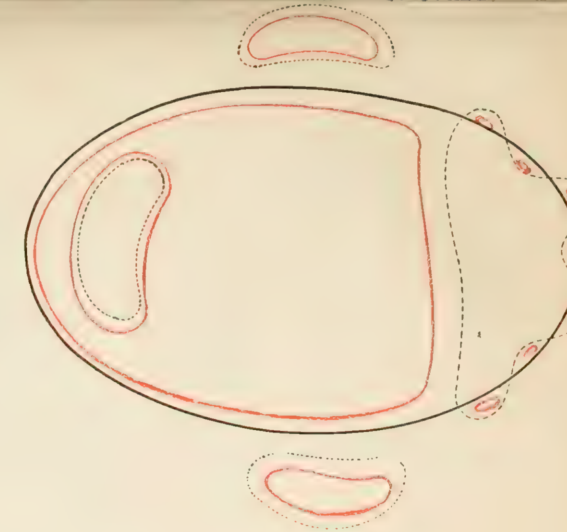
$$\underline{C_4} \equiv \underset{\cdots}{C_2} \cdot \underline{E_2} + \delta \prod_{i=1}^{i=4} l_i = 0.$$

FIG. 1.



$$\underline{C_4} \equiv \underset{\cdots}{C_2} \cdot \underline{E_2} + \delta \prod_{i=1}^{i=4} l_i = 0.$$

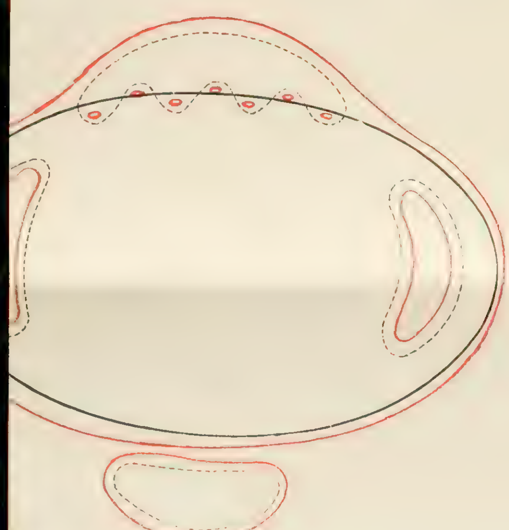
FIG. 2.

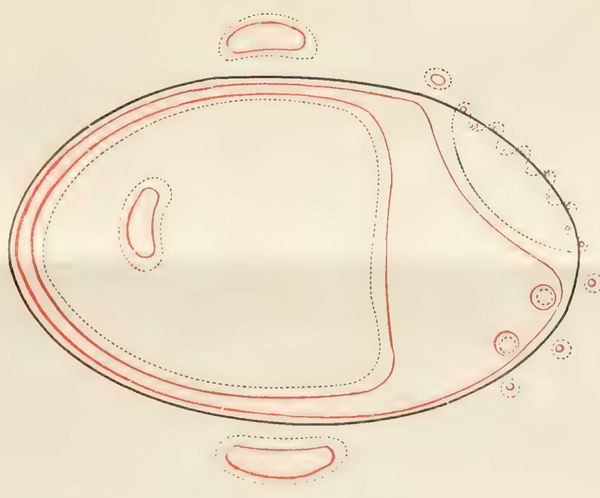


$$\underline{C_6} \equiv \underset{\cdots}{C_4} \; \underline{E_2} + \delta \prod_{i=1}^{i=6} l_i = 0.$$

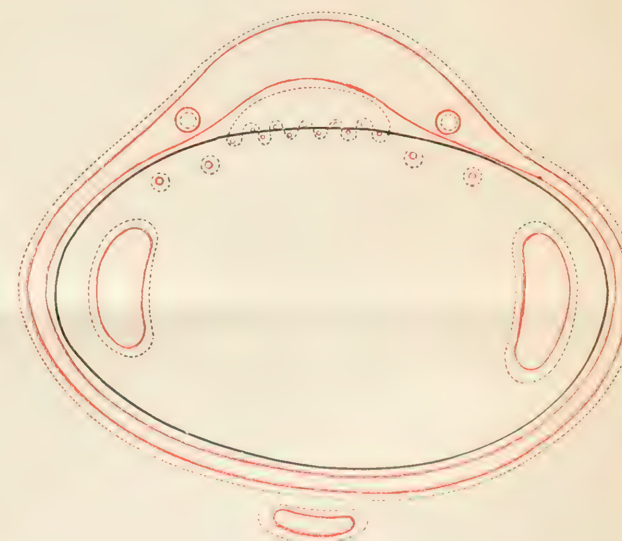
C_6 from C_4 by 1st mode of generation.

FIG. 3.





$i=8$



CORRESPONDENCES AND THE THEORY OF GROUPS*

BY

JOSEPH EDMUND WRIGHT

The object of this paper is to bring together several points connected with the general theory of correspondences and continuous groups, and to apply them to the theory of screws. Although the several results are in general not new, it seems of interest to give the accompanying presentation of the subject, as it furnishes an excellent example of the way in which the theory of continuous groups underlies the whole theory of correspondences.†

The first section is devoted to general theory. Use is made of the theorem of LIE‡ that if we have a continuous group in n variables together with an invariant equation system involving m parameters, then a 'group of the parameters' exists which is isomorphic with the given group, and it is pointed out that this theorem is fundamental in all correspondences.§ The correspondence established is that between a P_m and a P_n. Contact transformation is the particular case when $m = n$.

The screw geometry is developed from the projective group in three dimensions together with the system of equations which define a general straight line. The general theory leads at once to two important results in connection with the theory of groups:

1) *The general continuous conformal group in four dimensions is simply isomorphic with the general projective group in three.*

2) *Both these groups are simply isomorphic with the continuous projective group in five dimensions which leaves a given quadric invariant.*

There follows an immediate generalization of part of the second theorem. We have in fact the following:

3) *The general conformal group in space of n dimensions is simply isomorphic with the projective group in space of $n + 1$ dimensions which preserves a given quadric.*

These three results are due to KLEIN.‖ Some slight differences appear

* Presented to the Society December 29, 1905. Received for publication January 9, 1906.

† See KLEIN, *Erlangen Programme*, 1872; Bulletin of the American Mathematical Society, vol. 2 (1893), pp. 215 sqq.

‡ See *Continuirlichen Gruppen*, pp. 549, 718.

§ KLEIN, loc. cit.

‖ See in particular *Höhere Geometrie*, vol. 1, p. 487 sqq.

because we are concerned merely with *continuous* groups; for example, our conformal group does not include inversions. The theory leads naturally to KLEIN's* correspondence between linear complexes and hyperspheres, and the properties of that correspondence are developed by *a priori* reasoning. The sphere-straight line correspondence of LIE also appears in the natural course of development. The major part of the remainder of the paper is concerned with the correspondence shown to exist between points in space of five dimensions and screws. With respect to particular results we may mention that reciprocal screws become conjugate points with respect to the fundamental quadric, and that the process of finding the resultant of any number of wrenches on given screws is shown to be equivalent to the process of finding the mass centre of given masses at given points in the five dimensional space.

The following notation is used throughout: P_m denotes a linear m dimensional manifold in ordinary space of any dimensions; S_m denotes a hypersphere of m dimensions in ordinary space of any dimensions. A 'quadric' is a locus satisfying an equation of the second degree in any space.

§ 1.

Let

$$x'_i = f_i(x_1, \cdots, x_n, a_1, \cdots, a_r) \qquad (i=1, \cdots, n),$$

be the finite equations of a continuous group in the variables x, and let there be any equation system

$$\phi_s(x_1, \cdots, x_n, y_1, \cdots, y_m) = 0 \qquad (s=1, \cdots, p),$$

such that, for all transformations of the group, values $y'_1, \cdots, y'_m$ independent of x exist which make

$$\phi_s(x'_1, \cdots, x'_n, y'_1, \cdots, y'_m) = 0 \qquad (s=1, \cdots, p),$$

provided the ϕ's in the unaccented variables vanish. The general theory of groups shows that the transformations for the y's form a group which is isomorphic with the original one. This group need not necessarily be of the same order as the original one. Consider for example the general linear group

$$x_i = \sum_{j=1}^{n} \alpha_{ij} x_j \qquad (i=1, \cdots, n),$$

in conjunction with the equation system

$$x_i = y_i x_n \qquad (i=1, \cdots, n-1),$$

* KLEIN, Mathematische Annalen, vol. 5 (1872) p. 257; see also GRACE, Transactions Cambridge Philosophical Society, vol. 16 (1898), p. 153.

The group for the x's is of order n^2, whereas the group for the y's, the general projective group in $(n-1)$ dimensions, is only of order n^2-1.

Let Xf denote any infinitesimal operator of the x group, and Yf the corresponding operator of the y group, then the structure constants are the same for the two cases, but there may be linear relations between the operators of the y group. In any case the equations $\phi = 0$ form an invariant equation system for the group whose operators are $(X+Y)f$, and this group is isomorphic with both the x and y groups.

The general condition for simple isomorphism is readily obtained. Suppose that the y group is of order $r-h$, then there must exist h infinitesimal transformations of the x group which transform the system

$$\phi_s(x_1, \cdots, x_n, y_1, \cdots, y_m) = 0 \qquad (s=1, \cdots, p),$$

into the system

$$\phi_s(x'_1, \cdots, x'_n, y_1, \cdots, y_m) = 0.$$

There must thus exist a subgroup of the x group for which the equations $\phi = 0$ form an invariant system, for all values of the y's. For instance, in the case considered, $x'_i = \lambda x_i$ is such a subgroup of the general linear group.

Now consider any manifold of n dimensions and let any element in it be determined by $x_1, \cdots, x_n$. Consider also any manifold of m dimensions in which an element is determined by $y_1, \cdots, y_m$. Then we say that a correspondence exists between the two manifolds such that to an element of the first corresponds a certain locus in the second and vice versâ. We can say that the x and y groups correspond, and in fact the importance of the correspondence depends largely on these groups. For example, commencing with the ordinary projective group in space of three dimensions, we may take the equation $x_3 = x_1 y_1 + x_2 y_2 + y_3$ as the equation system $\phi = 0$. We thus establish a correspondence between points in the space y, and planes in the space x. Now planes in the space x passing through a given line are transformed by the x group into planes passing through another line. This condition must be an invariant one for points in the y space. We see therefore that points y lying on a line transform into points lying on a line. Also the cross ratio of four x planes through a given line is invariant under all transformations of the x group, and hence the cross ratio of four y points lying on a line is invariant under all transformations of the y group. The y group is in fact also projective. This idea may be immediately applied to the general case, and we see that any relation among different x loci gives an invariant relation among the corresponding y points, and conversely. It is immediately obvious that the correspondence considered is a generalization of a contact transformation. The Lie transformation of straight

lines into spheres gives a good idea of the importance of the underlying group. We mention two theorems.*

1. There are 15 infinitesimal contact transformations of spheres into spheres, and there are 15 infinitesimal transformations of straight lines into straight lines. Each of these sets forms a group and the two groups are simply isomorphic.

2. There are 10 infinitesimal point transformations of spheres, and there are ten infinitesimal transformations of straight lines into straight lines, which leave a given linear complex invariant. The two groups are again simply isomorphic.

§ 2.

We commence with the projective group in space. This group leaves invariant the family of all straight lines. We take for the equations $\phi = 0$ the two:

$$x_1 = y_1 x_3 + y_3, \qquad x_2 = y_2 x_3 + y_4.$$

This gives a correspondence between a manifold x of three dimensions and one y of four. To a line in the x space corresponds a point in the y space, and to a point in the x space corresponds a P_2 of a particular type in the y space. Corresponding to the projective group we have a 15 parameter group in four dimensions. The first preserves intersecting lines and hence there exists an invariant relation among two specially selected y points. If two lines y and y' intersect, then

$$(y_1 - y_1')(y_4 - y_4') - (y_2 - y_2')(y_3 - y_3') = 0.$$

Hence if we regard y as fixed, all the points on a certain hypersurface of the second order must transform into points on a similar hypersurface. It is at once seen that the above relation may be made symmetrical by taking instead of the y's themselves certain linear functions of them, namely,

$$y_1 = Y_1 + iY_2,\ y_4 = Y_1 - iY_2,\ y_2 = -Y_3 - iY_4,\ y_3 = -Y_3 + iY_4.$$

We have now the relation between two points in the Y space,

$$(Y_1 - Y_1')^2 + (Y_2 - Y_2')^2 + (Y_3 - Y_3')^2 + (Y_4 - Y_4')^2 = 0,$$

invariant under the transformations of a 15 parameter group. The 15 parameter group must therefore transform all points lying on a line which meets the sphere at infinity into points on another such line. It must therefore leave this sphere invariant. It must leave invariant the relation

$$dY_1^2 + dY_2^2 + dY_3^2 + dY_4^2 = 0;$$

*See LOVETT, Annali di Mathematica, ser. III[3], vol. 7 (1902), p. 39.

and therefore it must be such that

$$dY_1'^2 + dY_2'^2 + dY_3'^2 + dY_4'^2 = \rho(Y_1 Y_2 Y_3 Y_4)(dY_1^2 + dY_2^2 + dY_3^2 + dY_4^2),$$

where Y_1', Y_2', Y_3', Y_4' is the point obtained from Y_1, Y_2, Y_3, Y_4 by any operation of the group. The transformation must therefore be conformal. Conversely, if any infinitesimal conformal transformation be performed in the Y space it will give a transformation of lines into lines in the x space and all lines through a given point will transform into lines through a point. It will therefore be a point transformation which preserves straight lines and will thus be projective. Hence we have the theorem:

The general projective group in three dimensions and the general conformal group in four are simply isomorphic.

Further, it is easily seen that a hypersurface of the second order contains an infinite number of straight lines of which ∞^1 go through any point on the surface. Conversely this property defines a hypersurface of the second order. Now a hypersphere is such a surface and the lines in question are all minimal lines. Hence a hypersphere must transform into a hypersurface of the second order under the general conformal group. But this group leaves the sphere at infinity invariant. Hence the transformed surface must be a hypersphere. We see therefore that there must exist some complex of lines in three dimensional space which is transformed into a similar complex by all projections. This complex is given by

$$Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 + 2AY_1 + 2BY_2 + 2CY_3 + 2DY_4 + E = 0,$$

where A, B, C, D, E are constants. This becomes

$$(y_1 y_4 - y_2 y_3) + ay_1 + by_2 + cy_3 + dy_4 + e = 0,$$

where a, b, c, d, e are constants. It is in fact a linear complex. Hence we see that projection changes a linear complex into a linear complex, and further, this is the most general point transformation that will do so. Incidentally we note that a correspondence is established between a linear complex and a hypersphere.*

Now two S_3's have one common invariant under all conformal transformations, namely, their angle of intersection. Hence there is one invariant under all projections for two linear complexes. We shall find it convenient to speak, instead of the linear complex, of the *screw* for which the lines of the linear complex are null lines. Then a screw corresponds to an S_3.

* KLEIN, Mathematische Annalen, vol. 5 (1872), p. 264.

Let there be any wrench on a given screw, and consider the moment of the system round any line. If

$$S \equiv V_1^2 + V_2^2 + V_3^2 + V_4^2 + \cdots \text{etc.} = 0$$

be the S_3 corresponding to the given line, then the moment in question is λS, where λ is defined as the "intensity" of the wrench. Now consider any two screws; associate intensity $\lambda/(\lambda+\mu)$ with the first, $\mu/(\lambda+\mu)$ with the second, and combine. We get a unit wrench on a third screw which lies on the cylindroid determined by the two original screws. We conclude that to a system of screws on a cylindroid corresponds a system of S_3's having a common sphere. Such a coaxial system includes two points S_3's. Hence there are two screws of zero pitch on a cylindroid. The two points thus determined are inverse points with respect to any one of the linear system of S_3's. Now consider any S_3 containing both these points; it corresponds to a screw which has both these lines as null lines, and is therefore reciprocal to both the screws of zero pitch; hence it is reciprocal to all the screws of the cylindroid. But any S_3 through two inverse points of S_3' cuts S_3' orthogonally. Hence if the mutual invariant of two screws vanishes the two corresponding S_3's cut at right angles.

Now consider two P_2's in the four dimensional space; if every line in one is perpendicular to every line in the other, the two P_2's are said to be at right angles. Take any S_3 passing through a fixed S_2 in a fixed P_2, and let P_3 contain this P_2. All S_3's with their centres in P_3 will have their centres on a fixed line perpendicular to the P_2 and passing through the centre of the fixed circle. Hence all S_3's through the fixed circle will have their centres in a fixed P_2 perpendicular to the P_2 of the circle, and passing through its centre (the two P_2's have of course only one point common). There will thus be a locus in the second P_2 corresponding to the point spheres of the system. Two points of this locus lie in any P_3 through the given P_2. These points lie on the line through the centre of the fixed circle perpendicular to the P_2. They are equidistant from the centre, and this distance is independent of the particular P_3 selected. Hence the locus of point spheres of the system is another circle; the two circles have the same centre, and it is easy to see that the sum of the squares of the radii is zero. The relation between the two circles is reciprocal. The whole system of S_3's is determined by any three independent ones (i. e., ones not having a common S_2). If we translate these results into three dimensions we see that given three screws there is a single infinity of lines of the zero pitch belonging to the system thus determined. There exists a single infinity of screws of zero pitch reciprocal to the whole system. To the one system corresponds a circle, and to the other another circle which is definite when the first is given. We thus see that to a regulus corresponds a circle, and to the lines meeting the rays of a regulus correspond the points of another circle. Similar considerations may

be readily applied to the system arising from the four or five screws. Four S_3's have two points common, and the reciprocal system thus includes two point S_3's; this system is in fact coaxial. The five system consists of S_3's orthogonal to a given S_3. Now suppose we fix a given screw; by an inversion we may make the corresponding S_3 a P_3, and then a correspondence is established between the lines of a given complex and points of a P_3; for any point in the P_3 corresponds to a screw of zero pitch reciprocal to the given screw, that is to say, any point in the P_3 corresponds to a line of the complex determined by the fixed screw. Any S_3 meets the P_3 in a sphere, and for a given sphere there is a single infinity of S_3's. This single infinity is determined by any given S_3 and the P_3. It follows that we may regard a sphere as corresponding to all the screws of a cylindroid which contains the P_3. Now a cylindroid is determined uniquely by two of its screws, hence the P_3 and any line determine a unique sphere. But corresponding to any sphere there are two point S_3's, that is to say two lines, and hence the correspondence between lines and spheres is a $2-1$ correspondence. The points are images in the P_3, and therefore the lines are conjugates with respect to the given complex. Now consider any two spheres. They are the intersections of the P_3 with two point S_3's; the two spheres intersect in a circle, and therefore the two point S_3's lie on a circle which is perfectly determined if the first circle is given. The rays of a regulus correspond to the circle thus determined. The regulus is not general, for it is self conjugate with respect to the fundamental screw, and the directors are therefore null lines of this screw. Suppose that the two spheres touch; both circles now become point circles, and hence each of them is a pair of minimal lines. Hence if two spheres touch, the four corresponding four dimensional points lie on two minimal lines. As the spheres are supposed general, no two points corresponding to the same sphere lie on the same minimal line, and therefore the points lie by pairs on two minimal lines. Hence the corresponding lines in the three dimensional space intersect in pairs. We note that this also includes the theorem that if two straight lines intersect their conjugates intersect. This correspondence between lines and spheres is of course that of LIE. Without inversion we should have a correspondence between lines in ordinary space and spheres in elliptic space.

We may take six S_3's as coördinates, and thus express any point by means of its powers from these S_3's. The coördinates thus used are the equivalent in four dimensions of DARBOUX's * pentaspherical coördinates in three. As a particular simplification we take the S_3's to be mutually orthogonal, and we thus see that the method of discussing screws by referring them to six co-reciprocal coördinate screws is strictly in correspondence with DARBOUX's sphere geometry.

* See DARBOUX, *Théorie des surfaces*, vol. 1, p. 213.

We take the powers mentioned to be $z_1, z_2, z_3, z_4, z_5, z_6$, and then any S_3 is given by a linear relation among the z's. The condition for orthogonality of two S_3's

$$\lambda_1 z_1 + \lambda_2 z_2 + \lambda_3 z_3 + \lambda_4 z_4 + \lambda_5 z_5 + \lambda_6 z_6 = 0$$

and

$$\lambda_1' z_1 + \lambda_2' z_2 + \lambda_3' z_3 + \lambda_4' z_4 + \lambda_5' z_5 + \lambda_6' z_6 = 0$$

is

$$\lambda_1 \lambda_1' + \lambda_2 \lambda_2' + \lambda_3 \lambda_3' + \lambda_4 \lambda_4' + \lambda_5 \lambda_5' + \lambda_6 \lambda_6' = 0.$$

Hence if an S_3 cuts itself orthogonally,

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 + \lambda_5^2 + \lambda_6^2 = 0.$$

This is therefore the condition for a point S_3. If the point S_3 cuts

$$S_3 \equiv \sum_{r=1}^{6} \lambda_r' z_r = 0$$

orthogonally,

$$\lambda_1 \lambda_1' + \lambda_2 \lambda_2' + \lambda_3 \lambda_3' + \lambda_4 \lambda_4' + \lambda_5 \lambda_5' + \lambda_6 \lambda_6' = 0.$$

Hence the *point* $\lambda_1, \lambda_2, \cdots, \lambda_6$ lies on S_3, and this point is therefore the centre of the point S_3. Hence among the six coördinates of any point there exists the relation

$$\sum_{r=1}^{6} z_r^2 = 0. \dagger$$

§ 3.

Consider the general relation

$$Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 + 2X_1Y_1 + 2X_2Y_2 + 2X_3Y_3 + 2X_4Y_4 + X_5 = 0$$

in conjunction with the conformal group in the space Y. This equation is invariantive, and hence it establishes a correspondence between space of four and that of five dimensions. The group for the X's changes P_4's of a particular type into P_4's of the same type. Also the Y group transforms point S_3's into point S_3's, and therefore the X group must leave invariant the manifold

$$X_1^2 + X_2^2 + X_3^2 + X_4^2 = X_5.$$

Coaxial S_3's in the Y space correspond to lines in the X space, and thus lines are transformed into lines in the X space. The X group is thus seen to be projective and to keep a particular quadric invariant. Hence the projective group in five dimensions which preserves a particular quadric is isomorphic to the general projective group in three dimensions and to the

† In connection with this S_3 geometry see DARBOUX, loc. cit.

general conformal group in four. To a general line corresponds a coaxial system of S_3's, or, since the system is determined by its common sphere, the correspondence is one between lines in five and spheres in four dimensions. To a system of lines having a common point corresponds a system of spheres lying on a common S_3. To the points of a general P_2 correspond S_3's passing through a given circle, and hence we may say that to a given P_2 in five dimensions corresponds a circle in four. If two points are conjugate with respect to the quadric, the corresponding S_3's cut orthogonally. Hence to the points of a P_4 correspond all the S_3's cutting a given S_3 orthogonally. To the points of a P_3 correspond the S_3's orthogonal to two given S_3's, or passing through two given points, and so on. Comparing the five dimensionality with the three dimensionality with which we started, we see that, e. g., to straight lines correspond points on the fundamental quadric, and inversion with respect to a linear complex is equivalent to reflection of the quadric in a given point.

It is convenient to project the fundamental quadric into an S_4. This may be done by taking new coördinates $\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6$, where

$$\rho x_1 = \xi_1,\ \rho x_2 = \xi_2,\ \rho x_3 = \xi_3,\ \rho x_4 = \xi_4,\ \rho x_5 = -(\xi_5 + 1),\ \rho = (\xi_5 - 1).$$

The quadric now becomes

$$\sum_{r=1}^{6} \xi_r^2 = 1,$$

and the S_3 in the four dimensional space becomes

$$z_6 = \xi_1 z_1 + \xi_2 z_2 + \xi_3 z_3 + \xi_4 z_4 + \xi_5 z_5,$$

where the z's are a mutually orthogonal set of S's.

The plane representation for the three system of screws due to Sir ROBERT BALL * is seen to be a particular case of this correspondence; and we note that the projective group in space which projects a three system into itself is simply isomorphic with the plane projective group that preserves a given circle.

We can at once give a geometrical interpretation in five dimensions to any screw system. To a 2-system corresponds the system of points on a line, and in general to an n system corresponds the points of a P_{n-1}. The most general projection of an n system into itself corresponds to the most general projection of a P_{n-1} into itself which preserves a given S_{n-2}. For example, the most general projection of a cylindroid into itself corresponds to the homographic transformation of a straight line which preserves two given points and so on.

We note that corresponding to a P_n $(n = 0, \cdots, 4)$, in the five dimensionality there is a reciprocal P_{4-n} with respect to the fundamental quadric. Hence an n system of screws has a $6 - n$ system of reciprocal screws.

* *Theory of Screws* (1900). chap. 15.

Now consider n screws and let there be associated with them intensities $\lambda_1, \cdots, \lambda_n$. The resultant wrench will be of intensity $\lambda_1 + \cdots + \lambda_n$, and if $\xi_{r1}, \cdots, \xi_{r5}$ are the coördinates of the point corresponding to the rth screw, the point corresponding to the resultant screw has the coördinates

$$\sum_{s=1}^{n} \xi_{st}\lambda_s \Big/ \sum_{s=1}^{n} \lambda_s \qquad (t=1, \cdots, 5).$$

Hence finding the resultant wrench is equivalent to finding the mass centre of points ξ with associated multiples λ. The conditions for possibility of equilibrium of wrenches on n given screws follow at once from the five dimensional representation. In fact the corresponding points must obviously lie in a P_{n-1}. Hence the n screws must belong to an $n-1$ system.

BRYN MAWR,
January, 1906.

ON LAMÉ'S SIX EQUATIONS CONNECTED WITH TRIPLY ORTHOGONAL SYSTEMS OF SURFACES.

BY J. E. WRIGHT, M.A.

LAMÉ* has shown for a triply orthogonal system of surfaces given by the parameters ρ, ρ_1, ρ_2 that if the square of the element of length is given by $ds^2 = H^2 d\rho + H_1^2 d\rho_1^2 + H_2^2 d\rho_2^2$, where H, H_1, H_2 are certain functions of ρ, ρ_1, ρ_2, then H, H_1, H_2 must satisfy the following system of equations:

$$\frac{\partial^2 H}{\partial\rho_1 \partial\rho_2} = \frac{1}{H_1}\frac{\partial H}{\partial\rho_1}\frac{\partial H_1}{\partial\rho_2} + \frac{1}{H_2}\frac{\partial H}{\partial\rho_2}\frac{\partial H_2}{\partial\rho_1} \qquad (1)$$

and two others of the same type; (2), (3)

$$\frac{\partial}{\partial\rho_1}\left(\frac{1}{H_1}\frac{\partial H}{\partial\rho_1}\right) + \frac{\partial}{\partial\rho}\left(\frac{1}{H}\frac{\partial H_1}{\partial\rho}\right) + \frac{1}{H_2^2}\frac{\partial H}{\partial\rho_2}\frac{\partial H_1}{\partial\rho_2} = 0 \qquad (4)$$

with two others of this type. (5), (6)

Also if V is a function of x, y, z for which

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0,$$

he has shown that

$$\frac{\partial}{\partial\rho}\left(\frac{H_1 H_2}{H}\frac{\partial V}{\partial\rho}\right) + \frac{\partial}{\partial\rho_1}\left(\frac{H_2 H}{H_1}\frac{\partial V}{\partial\rho_1}\right) + \frac{\partial}{\partial\rho_2}\left(\frac{H H_1}{H_2}\frac{\partial V}{\partial\rho_2}\right) = 0.$$

If the system of coordinates ρ, ρ_1, ρ_2 is isothermal, this equation must be satisfied by $V = \rho$, or by $V = \rho_1$, or by $V = \rho_2$. Hence $H_1 H_2/H = Q^2$, where Q is a function of ρ_1 and ρ_2 only. Similarly $H_2 H/H_1 = Q_1^2$, and $HH_1/H_2 = Q_2^2$, where Q_i is a function not involving the variable ρ_i. Hence $H = Q_1 Q_2$, $H_1 = Q_2 Q$, $H_2 = Q Q_1$. The six equations given above transform into six in the variables Q. Lamé † gives a solution of

* Leçons sur les coordonées curvilignes (1859), pp. 76, 78.

† Loc. cit., p. 99.

these equations by first finding a solution of equations (1), (2), (3) and using this to solve the remainder. He makes the statement that his solution of the first three equations is the most general possible * ; this statement is obviously inaccurate and it seems of interest to give a complete solution of the equations. The solution of the first three equations, or rather a comparison of two different solutions, leads to a curious result in the theory of elimination.

Equation (1) becomes

$$Q\frac{\partial Q_2}{\partial \rho_1}\frac{\partial Q_1}{\partial \rho_2} = Q_1\frac{\partial Q_2}{\partial \rho_1}\frac{\partial Q}{\partial \rho_2} + Q_2\frac{\partial Q_1}{\partial \rho_2}\frac{\partial Q}{\partial \rho_1}, \tag{7}$$

with similar expressions for (2) and (3). Multiply (7) by $\partial Q_1/\partial \rho$, and the transformed expression for (3) by $\partial Q_1/\partial \rho_2$, and add. The result is

$$Q_1\left[\frac{\partial Q_2}{\partial \rho}\frac{\partial Q}{\partial \rho_1}\frac{\partial Q_1}{\partial \rho_2} + \frac{\partial Q_1}{\partial \rho}\frac{\partial Q_2}{\partial \rho_1}\frac{\partial Q}{\partial \rho_2}\right] = 0.$$

Hence unless all the Q's vanish identically

$$\frac{\partial Q_2}{\partial \rho}\frac{\partial Q}{\partial \rho_1}\frac{\partial Q_1}{\partial \rho_2} + \frac{\partial Q_1}{\partial \rho}\frac{\partial Q_2}{\partial \rho_1}\frac{\partial Q}{\partial \rho_2} = 0. \tag{8}$$

Assume that none of the derivatives in (8) vanish identically and write

$$K = -\frac{\partial Q}{\partial \rho_1}\Big/\frac{\partial Q}{\partial \rho_2}, \quad K_1 = -\frac{\partial Q_1}{\partial \rho_2}\Big/\frac{\partial Q_1}{\partial \rho}, \quad K_2 = -\frac{\partial Q_2}{\partial \rho}\Big/\frac{\partial Q_2}{\partial \rho_1}.$$

Equation (8) becomes $KK_1K_2 = 1$, where K_i is a function not involving ρ_i. By taking logarithms and differentiating with respect to two of the variables ρ, it is easy to prove that the most general values for the K's are

$$K = \frac{a_2}{a_1}, \quad K_1 = \frac{a}{a_2}, \quad K_2 = \frac{a_1}{a},$$

where a is a function of ρ only and similarly a_1 and a_2 are functions of ρ_1 and ρ_2 alone respectively.

* Loc. cit., p. 100, line 23.

If we put $\int a d\rho = \sigma$, with similar values for σ_1 and σ_2, we deduce that Q is a function of $\sigma - \sigma_2$ only, Q_1 a function of $\sigma_2 - \sigma$ only, and Q_2 a function of $\sigma - \sigma_1$ only. Substitute these values in (7) and it readily reduces to

$$\frac{Q}{Q'} + \frac{Q_1}{Q_1'} + \frac{Q_2}{Q_2'} = 0.$$

Hence

$$\frac{Q}{Q'} = n(\sigma_1 - \sigma_2), \quad \frac{Q_1}{Q_1'} = n(\sigma_2 - \sigma), \quad \frac{Q_2}{Q_2'} = n(\sigma - \sigma_1),$$

where n is a constant, and therefore

$$Q = c(\sigma_1 - \sigma_2)^{1/n}, \quad Q_1 = c_1(\sigma_2 - \sigma)^{1/n}, \quad Q_2 = c_2(\sigma - \sigma_1)^{1/n}, \quad (A)$$

where c, c_1, c_2 are constants. This is the most general solution, provided none of the derivatives in (8) vanish. If, however, for example, $\partial Q_2/\partial \rho_1 = 0$, then either $\partial Q_2/\partial \rho = 0$, or $\partial Q/\partial \rho_1 = 0$, or $\partial Q_1/\partial \rho_2 = 0$. Equation (7), however, shows that $\partial Q_2/\partial \rho_1 = 0$ implies either $\partial Q/\partial \rho_1 = 0$ or $\partial Q_1/\partial \rho_2 = 0$. $\partial Q_2/\partial \rho_1 = 0$, $\partial Q_1/\partial \rho_2 = 0$ lead to the solution

$$Q_1 = f(\rho), \quad Q_2 = cf(\rho), \quad Q = \phi(\rho_1 \, \rho_2), \qquad (B)$$

where f and ϕ are arbitrary functions of their arguments and c is a constant. $\partial Q_2/\partial \rho_1 = 0$, $\partial Q/\partial \rho_1 = 0$ lead to the solution

$$Q_2 = f(\rho), \quad Q = \phi(\rho_2), \quad Q_1 = F(Q, \, Q_2), \qquad (C)$$

where F is homogeneous and of unit degree in Q_1, Q_2. These types (A), (B), (C) are the only three types of solution.

We now proceed to the integration of the equations in a different manner. Write $\log Q = \lambda$, $\log Q_1 = \lambda_1$, $\log Q_2 = \lambda_2$. Equations (1), (2), (3) become

$$\frac{\partial \lambda_2}{\partial \rho_1}\frac{\partial \lambda_1}{\partial \rho_2} = \frac{\partial \lambda_2}{\partial \rho_1}\frac{\partial \lambda}{\partial \rho_2} + \frac{\partial \lambda}{\partial \rho_1}\frac{\partial \lambda_1}{\partial \rho_2} \qquad (9)$$

and two similar equations.

Write $\lambda_1 - \lambda_2 = \omega$, $\lambda_2 - \lambda = \omega_1$, $\lambda - \lambda_1 = \omega_2$ and (9) becomes

$$\frac{\partial\omega_1}{\partial\rho_1}\frac{\partial\omega_2}{\partial\rho_2} - \frac{\partial\omega_1}{\partial\rho_2}\frac{\partial\omega_2}{\partial\rho_1} = 0,$$

or

$$J\begin{pmatrix}\omega_1 & \omega_2\\ \rho_1 & \rho_2\end{pmatrix} = 0.$$

Hence the equation implies the existence of a relation, $f(\omega_1, \omega_2, \rho) = 0$. Exactly similarly there are relations

$$f_1(\omega_2, \omega, \rho_1) = 0, \quad f_2(\omega, \quad \omega_1, \rho_2) = 0.$$

From these equations, together with $\omega + \omega_1 + \omega_2 = 0$, it is easy to deduce that either ρ, ρ_1, ρ_2 do not any of them occur explicitly in f, f_1, f_2 or if, for example, ρ occurs in f, it is easy to prove that either ρ_1 is absent from f_1, or ρ_2 from f_2. The latter case is thus reduced to the former, for if one relation of the type $f(\omega, \omega_1) = 0$ exist, then two others of that type also exist in virtue of the relation $\omega + \omega_1 + \omega_2 = 0$.

Substituting for the ω's in terms of the Q's, we immediately deduce that the solution is equivalent to the statement that a homogeneous relation $F(Q, Q_1, Q_2) = 0$ exists among the Q's.

Combining the two solutions we have the following theorem connected with the theory of elimination:

Let $F(Q, Q_1, Q_2) = 0$ *be any homogeneous relation. It is possible to express* Q *as a function of two variables* ρ_1 *and* ρ_2, Q_1 *as a function of two variables* ρ_2 *and* ρ, *and* Q_2 *as a function of the variables* ρ *and* ρ_1, *in two cases only*:

(*A*) *If* F *is of the form* $aQ^n + a_1 Q_1^n + a_2 Q_2^n$, *where* a, a_1, a_2 *are constants.**

(*C*) *If* F *is general, and e. g.* Q *is a function of* ρ_1 *only, and* Q_1 *a function of* ρ *only.*

It is not difficult to complete the solution of the equations (1), $\cdots$, (6). It may readily be shown that for case (*A*) (4), (5), and (6) are not satisfied unless $n = \frac{1}{2}$, and then

$$\sigma = A\wp\left(\frac{\rho + a}{\sqrt{c}}, g_2, g_3\right) + B,$$

$$\sigma_1 = A\wp\left(\frac{\rho_1 + a_1}{\sqrt{c_1}}, g_2, g_3\right) + B,$$

* This includes (*B*).

$$\sigma_2 = A\wp\left(\frac{\rho_2 + a_2}{\sqrt{c_2}}, g_2, g_3\right) + B,$$

where the a's, the g's, A and B are constants.

For case (c) the complete solution is

$$Q_1 = a \operatorname{cosec}(b\rho_2 + c), \quad Q_2 = a' \operatorname{cosec}(b'\rho_1 + c'),$$
$$Q = A[\operatorname{cosec}^2(b\rho_2 + c) - \operatorname{cosec}^2(b'\rho_1 + c')]^{\frac{1}{2}},$$

where $A, a, a'; b, b', c, c'$ are constants such that

$$a^2b'^2 + a'^2b^2 = 0.$$

In case (B)

$$Q_1 = \frac{1}{a\rho + b}, \quad Q_2 = \frac{k}{a\rho + b},$$

where a, b, k are constants, and Q is a function of ρ_1 and ρ_2 which satisfies the equation

$$\left(k^2\frac{\partial^2}{\partial\rho_2^2} + \frac{\partial^2}{\partial\rho_1^2}\right)\log Q + a^2Q^2 = 0.$$

Of these three types of solution, the first is the same as that given by Lamé.* He gives it in different form, and his method of obtaining it is different. He falls into the error of imagining that the most general solution of the *first* three of his equations corresponds to the case of $n = \frac{1}{2}$, and it happens that this error is largely corrected because the *second* three equations require this limitation in case (A); case (B), however, escapes his notice.

The surfaces corresponding to the three solutions are readily obtained. (A) gives a system of confocal quadrics, and (C) a system of confocal spheroids with their axial planes. (B) gives a system of concentric spheres, and the conical surfaces obtained by joining the common centre to any set of isothermal lines on one of them.

Bryn Mawr,
October, 1905.

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* Loc. cit., p. 104.

NOTE ON REGULAR POLYGONS*

By Charlotte Angas Scott

It is a well-known fact that the circle described with its centre at any point P of a rectangular hyperbola and passing through the other extremity of the diameter through P cuts the hyperbola again at the vertices of an equilateral triangle. The object of the present note is to show that the circles which cut a rectangular hyperbola at four of the vertices of a regular pentagon, heptagon, or nonagon can be described with equal facility.

It is necessary, in the first place, to find a rectangular hyperbola that shall cut a given circle, $x^2 + y^2 = 1$, at four vertices of one of these polygons.

1. The Heptagon. If a side of the heptagon subtends at the centre an angle θ, then $7\theta = 2\pi$, hence $\sin 3\theta = -\sin 4\theta$; that is,

$$3 \sin \theta - 4 \sin^3 \theta = -4 \sin \theta \cos \theta \ (2 \cos^2 \theta - 1),$$

from which,

$$\sin \theta \ \{8 \cos^3 \theta + 4 \cos^2 \theta - 4 \cos \theta - 1\} = 0.$$

If one vertex is at $(1, 0)$ (see figure 1) the roots of this equation give the values of θ for the vertices. If x is written for $\cos \theta$, the solution $\sin \theta = 0$, which refers to the vertex $(1, 0)$ and gives therefore $\theta = 0$, becomes $x - 1 = 0$; and the cubic factor gives

$$8x^3 + 4x^2 - 4x - 1 = 0.$$

Hence the roots of

$$(x - 1)(8x^3 + 4x^2 - 4x - 1) = 0,$$

that is, of

$$8x^4 - 4x^3 - 8x^2 + 3x + 1 = 0,$$

are the abscissæ of the vertices numbered 7, 1 and 6, 2 and 5, 3 and 4, in figure 1.

It is desired to find a rectangular hyperbola that shall meet the circle at the point 7, and at one point of every pair (1, 6), (2, 5), (3, 4). There are clearly 2^3, that is, 8 such rectangular hyperbolas; any one will serve the purpose.

* Presented to the American Mathematical Society, October 27, 1906.

The general equation of a rectangular hyperbola is

$$ax^2 + 2hxy - ay^2 + 2gx + 2fy + c = 0.$$

To find where this meets the circle

$$x^2 + y^2 - 1 = 0,$$

eliminate y. The two equations give, by elimination of the term y^2,

$$2ax^2 + 2hxy + 2gx + 2fy + c - a = 0,$$

that is,

$$2ax^2 + 2gx + c - a = -y\,(2hx + 2f)\,;$$

hence, since

$$-y^2 = x^2 - 1,$$

$$(x^2 - 1)(2hx + 2f)^2 + (2ax^2 + 2gx + c - a)^2 = 0.$$

The equation for the abscissæ of the common points is therefore

$$(4h^2 + 4a^2)\,x^4 + (8hf + 8ag)\,x^3 + \{4f^2 - 4h^2 + 4g^2 + 4a(c - a)\}\,x^2$$
$$+\{-8hf + 4g(c - a)\}\,x - 4f^2 + (c - a)^2 = 0,$$

and this is to be the same as

$$8x^4 - 4x^3 - 8x^2 + 3x + 1 = 0.$$

Hence, equating coefficients,

$$4h^2 + 4a^2 = 8\lambda, \qquad (1)$$

$$8hf + 8ag = -4\lambda, \qquad (2)$$

$$4f^2 - 4h^2 + 4g^2 + 4a\,(c - a) = -8\lambda, \qquad (3)$$

$$-8hf + 4g\,(c - a) = 3\lambda, \qquad (4)$$

$$-4f^2 + (c - a)^2 = \lambda. \qquad (5)$$

The sum of these gives

$$4g^2 + 4g(c + a) + (c + a)^2 = 0,$$

that is, $$2g + c + a = 0.$$

The sum of (2) and (4) gives

$$4g(c + a) = -\lambda.$$

Hence
$$-2g = a + c, \qquad \lambda = 2(a + c)^2.$$

With these values for g and λ, (1), (5), (2) become

$$h^2 = 4(a + c)^2 - a^2,$$

$$4f^2 = (c - a)^2 - 2(a + c)^2 = -(a + c)^2 - 4ac,$$

$$2hf = a(a + c) - 2(a + c)^2.$$

Hence, in virtue of the identity $h^2 \times 4f^2 = (2hf)^2$, we have

$$\{4(a + c)^2 - a^2\}\{-(a + c)^2 - 4ac\} = (a + c)^2\{a - 2(a + c)\}^2.$$

Since $2(a + c) - a$ is a factor of each side of this equation, one solution is $2(a + c) - a = 0$, that is,

$$a = -2c.$$

This leads to
$$2g = -(a + c) = c,$$

$$\lambda = 2(a + c)^2 = 2c^2,$$

$$h^2 = 4(a + c)^2 - a^2 = 4c^2 - 4c^2 = 0,$$

$$4f^2 = -(a + c)^2 - 4ac = -c^2 + 8c^2 = 7c^2.$$

Since c is a factor in all the coefficients, it is not zero; hence we may give it any convenient value, for example, -1. Then $a = 2$, $h = 0$, $2g = -1$, $2f = \pm\sqrt{7}$; choose for $2f$ the value $+\sqrt{7}$.

The four possible values for $a : c$, and the two for f, account for the eight solutions of the problem. The other three values for $a : c$, given by the cubic

$$\{2(a + c) + a\}\{-(a + c)^2 - 4ac\} = (a + c)^2\{2(a + c) - a\},$$

are irrational. Since only one solution out of the eight is needed, it is unnecessary to attend to these.

The rectangular hyperbola is

$$2x^2 - 2y^2 - x + \sqrt{7}\,y - 1 = 0.$$

The center is at $(\frac{1}{4}, \frac{1}{4}\sqrt{7})$; when the origin is transferred to this point the equation becomes

$$x^2 - y^2 = \frac{1}{8},$$

hence the semi-axis major is $\frac{1}{4}\sqrt{2}$. The curve lies as shown in figure 1; it passes through the vertices 1, 2, 4, 7.

Since the foci of $x^2 - y^2 = \frac{1}{8}$ are $(\pm\frac{1}{2}, 0)$, with the original axes they are $(\frac{3}{4}, \frac{1}{4}\sqrt{7})$, $(-\frac{1}{4}, \frac{1}{4}\sqrt{7})$; of these the first lies on the circle. The corre-

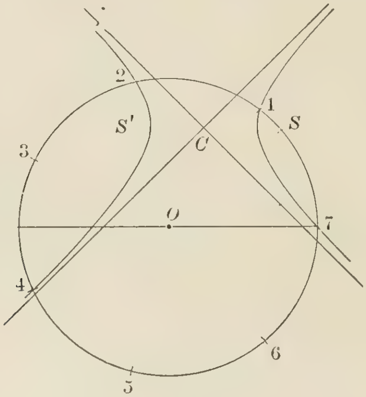

FIG. 1.

sponding directrices are $x = \pm\frac{1}{4}$, that is, referred to the original axes, $x = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$, and $x = -\frac{1}{4} + \frac{1}{4} = 0$; the second of these passes through the centre of the circle. The distance between the foci is equal to the radius of the circle.

2. The Nonagon. In the case of the nonagon, the angle θ is determined by $3\theta = \frac{2}{3}\pi$; hence $\cos 3\theta = -\frac{1}{2}$, that is,

$$4\cos^3\theta - 3\cos\theta + \tfrac{1}{2} = 0,$$

or

$$8x^3 - 6x + 1 = 0.$$

The roots of this equation are the abscissæ of the vertices 1, 4, 7 (see figure 2), hence of the pairs of vertices (1, 8), (4, 5), (7, 2). To bring in 3 and 6, for which $x = -\frac{1}{2}$, we multiply by $2x + 1$; the equation becomes

$$16x^4 + 8x^3 - 12x^2 - 4x + 1 = 0.$$

By the same argument as in the case of the heptagon, the coefficients in

the equation of the desired rectangular hyperbola are determined by the five equations

$$
\begin{aligned}
4h^2 + 4a^2 &= 16\lambda, &(1)\\
8hf + 8ag &= 8\lambda, &(2)\\
4f^2 - 4h^2 + 4g^2 + 4a(c-a) &= -12\lambda, &(3)\\
-8hf + 4g(c-a) &= -\lambda, &(4)\\
-4f^2 + (c-a)^2 &= \lambda. &(5)
\end{aligned}
$$

These, combined exactly as before, give

$$4g^2 + 4g(c+a) + (c+a)^2 = 9\lambda,$$

$$4g(c+a) = 4\lambda,$$

hence $$4g^2 - 5g(c+a) + (c+a)^2 = 0,$$

that is, $$\{4g - (c+a)\}\{g - (c+a)\} = 0$$

from which $$4g = c + a \quad \text{or} \quad g = c + a.$$

Of these, $4g = c + a$ leads to the simplest results; we find $4\lambda = (c+a)^2$, and then equations (1), (5), (2) become

$$h^2 = (a+c)^2 - a^2 = c(2a+c),$$

$$16f^2 = 4(c-a)^2 - (a+c)^2 = (a-3c)(3a-c),$$

$$4hf = (a+c)^2 - a(a+c) = c(a+c).$$

Hence, by means of $h^2 \times 16f^2 = (4hf)^2$,

we have

$$c(2a+c)(a-3c)(3a-c) = c^2(a+c)^2.$$

One solution of this equation is $c = 0$; the other three are irrational. We take therefore $c = 0$; then $h = 0$, $4g = a$, $16f^2 = 3a^2$, from which $4f = \pm\sqrt{3}\,a$; take $4f = +\sqrt{3}\,a$.

(Since there are four pairs of vertices, the number of solutions to be expected is 2^4, that is 16. All are accounted for; two values for g, four for $a : c$, two for f, give $2 \times 4 \times 2 = 16$ solutions).

Since a is a factor in all the coefficients, it is not zero; hence we may assign any convenient value, in this case 2. The desired rectangular hyperbola is

$$2x^2 - 2y^2 + x + \sqrt{3}\,y = 0.$$

The centre is at $(-\frac{1}{4}, \frac{1}{4}\sqrt{3})$; when the origin is transferred to this point the equation becomes

$$-x^2 + y^2 = \frac{1}{8},$$

hence the semi-axis major is $\frac{1}{4}\sqrt{2}$. The curve lies as shown in figure 2 ; it passes through the vertices 2, 3, 5, 8, and through the centre of the circle.

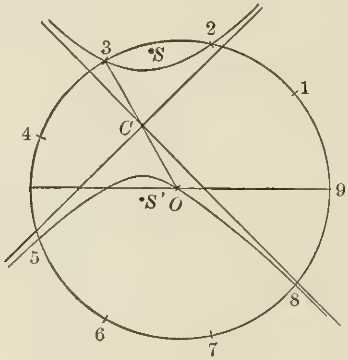

FIG. 2.

The radius of the circle to the vertex 3 is a diameter of the hyperbola; the distance between the foci is equal to the radius of the circle.

The rectangular hyperbola thus obtained for the nonagon is of the same size as that obtained for the heptagon, but differently placed.

3. The Pentagon. In the case of the pentagon, $5\theta = 2\pi$, hence $\sin 2\theta = -\sin 3\theta$,

from which

$$2 \sin\theta \cos\theta + 3 \sin\theta - 4 \sin^3\theta = 0 ;$$

that is,

$$\sin\theta = 0 \quad \text{or} \quad 2 \cos\theta + 3 - 4 \sin^2\theta = 0.$$

Hence
$$4 \cos^2\theta + 2 \cos\theta - 1 = 0,$$

that is,
$$4x^2 + 2x - 1 = 0.$$

The roots of this quadratic are the abscissæ of the pairs of vertices (1, 4), (2, 3) see figure 3. A rectangular hyperbola, with the axis of x as one axis of symmetry, can be passed through the four points. The equation of such a hyperbola is

$$ax^2 - ay^2 + 2gx + c = 0 ;$$

the abscissæ of the intersections with the circle are the roots of

$$ax^2 + a(x^2 - 1) + 2gx + c = 0,$$

that is, of

$$2ax^2 + 2gx - (a - c) = 0.$$

Hence $2a = 4$, $2g = 2$, $a - c = 1$, and the hyperbola is therefore

$$2x^2 - 2y^2 + 2x + 1 = 0.$$

The centre is at $(-\frac{1}{2}, 0)$; when this point is taken as origin the equation becomes

$$x^2 - y^2 + \tfrac{1}{4} = 0,$$

hence the semi-axis major is $\frac{1}{2}$. The curve lies as shown in figure 3. The radius of the circle to the point $(-1, 0)$ is the minor axis of the hyperbola.

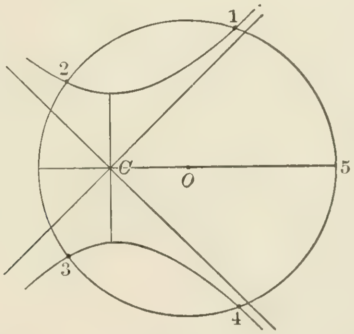

FIG. 3.

4. Conclusion. If we change the origin in all three cases to the centre of the hyperbola, and change the scale so that the hyperbola has the semi-axis major $= 1$; and moreover, in the case of the heptagon, interchange the axes of x and y; we arrive at the conclusion that the hyperbola $x^2 - y^2 + 1 = 0$ is cut at four vertices of a regular pentagon by the circle $(x - 1)^2 + y^2 = 4$; at four vertices of a regular heptagon by the circle

$$\left(x - \sqrt{\frac{7}{2}}\right)^2 + \left(y + \frac{1}{\sqrt{2}}\right)^2 = 8;$$

at four vertices of a regular nonagon by the circle

$$\left(x - \frac{1}{\sqrt{2}}\right)^2 + \left(y + \sqrt{\frac{3}{2}}\right) = 8.$$

The relation of the different circles to the hyperbola is, however, more neatly expressed by the geometrical statements of the results :

(1) A rectangular hyperbola is cut at four vertices of a regular pentagon by a circle on the minor axis as a radius (figure 4).

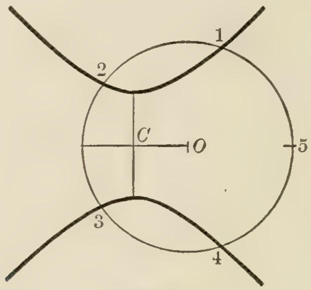

FIG. 4.

(2) A rectangular hyperbola is cut at four vertices of a regular heptagon by a circle of radius equal to the distance between the foci, with its centre on one directrix and passing through the other focus (figure 5).

(3) A rectangular hyperbola is cut at four vertices of a regular nonagon by a circle which has for radius a diameter of the hyperbola of length equal to the distance between the foci (figure 6).

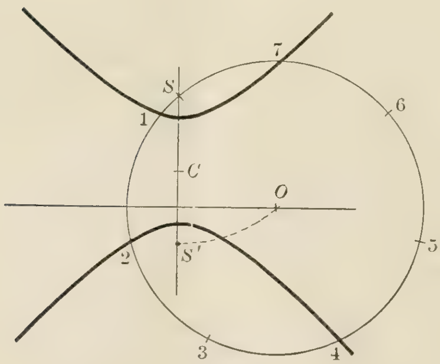

FIG. 5.

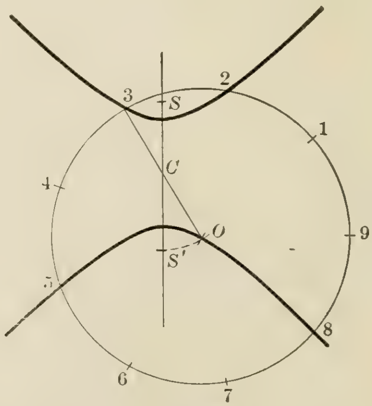

FIG. 6.

BRYN MAWR COLLEGE,
BRYN MAWR, PENNSYLVANIA,
OCTOBER, 1906.

NODAL CUBICS THROUGH EIGHT GIVEN POINTS

By J. E. WRIGHT.

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IN the pencil of cubics that passes through eight given points there are in general twelve with nodes. In certain special cases there may be two or more on one particular cubic of the pencil, and then that one is reducible. If, for instance, two lie on the same cubic, that cubic consists of a straight line and a conic. In other cases the number of nodal cubics may be reduced, due to coincidences of nodes. We wish to consider particularly the general pencil from which these particular cases are excluded. The questions with which we are concerned as regards their nodes are those of *analysis situs*; if a certain number of the eight points are real, we wish to know how many of the nodes are real crunodes, how many real acnodes, and how many imaginary. The case that lends itself most easily to algebraic treatment is that where all the eight base points, and therefore the ninth, are real, but it will be obvious from the first portion of the paper that pure algebra involves too heavy work for one to hope for much progress along that line. The results obtained are got by first mapping the plane containing the cubics on a cubic surface in space, by then projecting this cubic surface on a plane from a point on it, and finally by consideration of the tangents of the quartic thus obtained.

First, however, we consider a purely algebraic method in the case when the nine base points are real. Assume in this case that the vertices of the fundamental triangle are nodes of three of the nodal cubics through nine real points. These cubics are

$$C_1 \equiv (a_0 a_1 a_2 a_3 \between yz)^3 + 3\,(b_0 b_1 b_2 \between yz)^2\, x = 0,$$

$$C_2 \equiv (a'_0 a'_1 a'_2 a'_3 \between zx)^3 + 3\,(b'_0 b'_1 b'_2 \between zx)^2\, y = 0,$$

$$C_3 \equiv (a''_0 a''_1 a''_2 a''_3 \between xy)^3 + 3\,(b''_0 b''_1 b''_2 \between xy)^2\, z = 0,$$

and since they have nine points common we may take, without loss of generality

$$C_1 + C_2 + C_3 \equiv 0.$$

It is immediately clear from this condition that if we write

$$P \equiv x\,(a_1 x + b_1 y + c_1 z) + m_1 yz,$$
$$Q \equiv y\,(a_2 x + b_2 y + c_2 z) + m_2 zx,$$
$$R \equiv z\,(a_3 x + b_3 y + c_3 z) + m_3 xy,$$

and if we change the notation, we may write C_1, C_2, C_3, in the form

$$C_1 \equiv yQ - zR, \qquad C_2 \equiv zR - xP, \qquad C_3 \equiv xP - yQ.$$

At a common point of C_1, C_2, C_3, we have

$$xP = yQ = zR.$$

We assume that this point (x, y, z) is not on a side of the fundamental triangle, and write

$$xP = yQ = zR = \lambda xyz.$$

The value of λ is then determined by eliminating x, y, z from $P = \lambda yz$, $Q = \lambda zx$, $R = \lambda xy$ and solving the equation thus obtained. This equation is of the ninth order and to a real value of λ corresponds a real point (x, y, z) and to a real point (x, y, z) a real value of λ. Hence, if all the intersections of C_1, C_2, C_3 are real, all the roots of the equation in λ are real. We proceed to find this equation.

Firstly,
$$x\,(a_1 x + b_1 y + c_1 z) + (m_1 - \lambda)\,yz = 0, \tag{1}$$
$$y\,(a_2 x + b_2 y + c_2 z) + (m_2 - \lambda)\,zx = 0, \tag{2}$$
$$z\,(a_3 x + b_3 y + c_3 z) + (m_3 - \lambda)\,xy = 0. \tag{3}$$

From (2) and (3),

$$(a_2 x + b_2 y + c_2 z)(a_3 x + b_3 y + c_3 z) = (m_2 - \lambda)(m_3 - \lambda)\,x^2, \tag{4}$$

and there are two equations similar to this.

Write $m_1 - \lambda = \lambda_1$, ..., and assume $m_1 + m_2 + m_3 = 0$. This assumption involves no loss of generality. (4) becomes

$$(a_2 a_3 - \lambda_2 \lambda_3)\,x^2 + b_2 b_3 y^2 + c_2 c_3 z^2 + (b_2 c_3 + b_3 c_2)\,yz + (c_2 a_3 + c_3 a_2)\,zx + (a_2 b_3 + a_3 b_2)\,xy = 0.$$

By means of (2) and (3) this may be reduced to

$$(a_2 a_3 - \lambda_2 \lambda_3)\,x^2 - b_3 \lambda_2 zx - c_2 \lambda_3 xy + (b_2 c_3 - b_3 c_2)\,yz + c_3 a_2 zx + a_3 b_2 xy = 0,$$

or, from (1),

$$(a_2 a_3 - \lambda_2 \lambda_3)\left\{\frac{-\lambda_1 yz - b_1 xy - c_1 xz}{a_1}\right\} - (b_3 \lambda_2 - c_3 a_2)\,zx - (c_2 \lambda_3 - a_3 b_2)\,xy + (b_2 c_3 - b_3 c_2)\,yz = 0,$$

or
$$\{(a_2a_3-\lambda_2\lambda_3)\lambda_1-(b_2c_3-b_3c_2)a_1\}yz$$
$$+\{(a_2a_3-\lambda_2\lambda_3)c_1+a_1(b_3\lambda_2-c_3a_2)\}zx$$
$$+\{(a_2a_3-\lambda_2\lambda_3)b_1+a_1(c_2\lambda_3-a_3b_2)\}xy=0.$$

From this and two similar equations we have a three row determinant equated to zero as the equation for λ, by eliminating yz, zx, xy.

This equation when multiplied out is

$$-\lambda_1^3\lambda_2^3\lambda_3^3+2\lambda_1^2\lambda_2^2\lambda_3^2(a_2a_3\lambda_1+b_3b_1\lambda_2+c_1c_2\lambda_3)$$
$$+\text{terms involving } \lambda^6 \text{ and lower powers of } \lambda=0,$$

or
$$\lambda^9+\{3(m_2m_3+m_3m_1+m_1m_2)-2(a_2a_3+b_3b_1+c_1c_2)\}\lambda^7$$
$$+\text{lower powers of } \lambda=0.$$

Since this equation has all its roots real, by Sturm's theorem

$$3(m_2m_3+m_3m_1+m_1m_2)-2(a_2a_3+b_3b_1+c_1c_2) \text{ is negative.}$$

Now $C_1\equiv y^2(a_2x+b_2y+c_2z)-z^2(a_3x+b_3y+c_3z)+(m_2-m_3)xyz=0$ has a crunode or an acnode at $y=0$, $z=0$, according as $(m_2-m_3)^2$ is greater or less than $-4a_2a_3$. If the three nodes considered are all acnodes

$$(m_2-m_3)^2+(m_3-m_1)^2+(m_1-m_2)^2 \text{ is } < -4(a_2a_3+b_3b_1+c_1c_2).$$

Now $\Sigma(m_2-m_3)^2=2\Sigma m_1^2-2\Sigma m_2m_3=2(\Sigma m_1)^2-6\Sigma m_2m_3=-6\Sigma m_2m_3.$

Hence for three acnodes

$$3\Sigma m_2m_3 \text{ is } > 2(a_2a_3+b_3b_1+c_1c_2).$$

But this is impossible, by the result just obtained. Hence there cannot be more than two real acnodes. We therefore have the theorem—

If a pencil of cubics have nine real points common, it cannot include more than two acnodal cubics.

As further progress on these lines seems difficult, we start afresh and consider four independent cubics through six of the points. We use Clebsch's transformation* by means of the linear system thus determined, that is to say, if C_1, C_2, C_3, C_4 are the four independent cubics we take coordinates in space x_1, x_2, x_3, x_4 proportional to C_1, C_2, C_3, C_4.

The points in the plane are now mapped on a cubic surface in space, and the correspondence between the two is (1, 1) and is real. The correspondence breaks down for each of the six base points, and each of

* Clebsch, *Crelle's Journal*, Vol. LXV. (1866), p. 359.

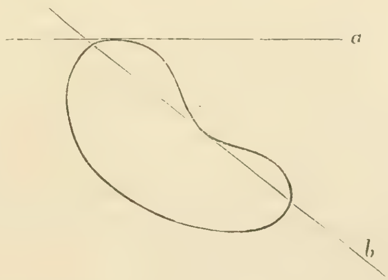
these corresponds to a real straight line on the surface. The other 21 lines on the surface are in this case all real, and six of them correspond to the six conics through five of the six base points. The remaining 15 correspond to the 15 lines joining the six points in pairs. If 2, 4, or 6 of the base points are pairs of conjugate imaginaries, the correspondence exists as before, and in these cases it is clear that the cubic surface has on it 15, 7, or 3 real straight lines respectively. Further, if any non-singular cubic surface is given, a real transformation of this kind can always be found between it and a plane, provided the surface contains two real non-intersecting straight lines. The only case of failure arises therefore when all the real straight lines on the surface lie in a plane. This case can only occur when there are not more than three such lines and these lie in a plane. Hence incidentally we see that of the lines on a cubic surface either 27, 15. 7, or 3 or less are real. It is clear that in the cases where the correspondence can be established the cubic surface has only one sheet (is unipartite) for the correspondence with the plane is (1, 1) and is real. But, obviously, bipartite cubic surfaces can exist, and hence if such a surface be bipartite it can have at most three real lines on it and these must lie in a plane. There are, of course, some surfaces with only three lines on them, lying in a plane, for which the correspondence exists, and these are unipartite. It does not follow that the correspondence can be established for every such unipartite surface. [This is, however, true.—*January 4th*, 1908.]

Now consider any cubic of the pencil. It corresponds to a section of the cubic surface by a plane. Two cubics intersect in three points outside the base points, and these clearly correspond to the three points in which the common line of the two planes in space meets the cubic surface. Hence the pencil of cubics through nine real points corresponds to sections of the surface by planes through a line which meets the surface in three real points. If only seven base points are real the correspondence is of the same kind, except that the axis of the planes meets the surface in one real point.

Now, take one of the three points on the axis and project the surface on to any plane. The result will be that the plane is divided into portions, for one of which all projectors meet the surface in two other real points, while for the remainder the projectors meet the surface in no other real points.

The bounding curve is known to be a quartic. Now consider any nodal cubic in the original plane. It is clear from the nature of the correspondence that the plane section of the cubic surface will also be a nodal cubic, and further that a crunodal cubic will give a crunodal cubic, and an

acnodal an acnodal. The nodal cubics thus correspond to the tangent planes through the given line, and thus finally to tangents to the quartic through a given point. Now the lines on the cubic surface are known to be double tangents of the quartic, and there is one other which is always real. Further, a quartic curve possesses four and only four double tangents called by Zeuthen *of the first kind*, and the remainder are of the second kind. Those of the second kind touch two different branches of the quartic, and have, of course, real contacts. The four of the first kind either have imaginary contacts or their two contacts are on the same branch of the quartic. Now each pair of external ovals of the quartic gives rise to four double tangents of the second kind, and hence the quartic has 1, 2, 3, or 4 external ovals according as the cubic surface has 3, 7, 15, or 27 real lines on it. Hence in the case we are considering at present the quartic has four ovals, all external. Hence, if we imagine an eye placed at the centre of projection of the cubic surface, the surface will appear to have four holes through it. If one hole be that given, a tangent plane such as a will obviously give rise to a crunodal section, whilst one



such as b will give rise to an acnodal section. Also the point of intersection of the tangents is certainly in the region external to the four ovals.

The following results are immediately obvious :—

(1) There are eight tangents of the type a, two to each oval.

(2) There may be one or two tangents of the type b, but in this case the quartic must have at least two or four inflexions.

(3) Each tangent of the type b carries an additional tangent of the type a, for then four tangents may be drawn from the point to that oval.

(4) As there cannot be more than twelve tangents altogether, there cannot be more than two of the type b.

The case when the axis of the planes meets the cubic surface in one real point may be similarly discussed. In this case the point from which

the tangents to the quartic are drawn is inside one of the ovals. The cases now are—

(5) Six of type a.

(6) One of type b and seven of type a.

(7) Two of type b and eight of type a, and this case implies the existence of two inflexions on the oval in which the point lies.

(8) Possibly three of type b and nine of type a, and the existence of four inflexions is implied on the oval in which the point lies.

By taking two of the six original base points as a pair of conjugate imaginaries we may treat this case by means of a quartic with three ovals and a point external to them. In a similar manner we may deal with cases where 4, 6, or 8 of the intersections of a pencil of cubics are imaginary, by means of quartics with three, two, or one external ovals, and we have the following final results.

Let R denote the number of real base points, C the number of real crunodes, A the number of real acnodes, then

(i.) $$C-A = R-1.$$

(ii.) *If R is 9, A may be 1 or 2.*

(iii.) *If R is > 1, the question is exactly the same as that of the number of real tangents that may be drawn to a quartic with 2, 3, or 4 ovals.*

(iv.) *If $R = 1$, the number of real crunodes is equal to the number of real acnodes, but we cannot give an upper or lower limit to the number by this method.*

The correspondence established leads to some other results. We see that if two of the nine base points of two cubics coincide, we have a point on the quartic as the point from which tangents are drawn, and it follows that the node at the double base point counts for two in the twelve. Further, a cusp arises for an inflexional tangent, and therefore a cusp counts for two of the twelve nodes. It is also clear that a cusp arises from the coincidence of a crunode and an acnode.

[*Added January 4th*, 1908.—In view of the addition on p. 55, it is clear that, if $R = 1$, the possible number of real double points is the same as the possible number of real tangents that can be drawn to a quartic consisting of a single oval, from a point inside that oval.]

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DOUBLE POINTS OF UNICURSAL CURVES.

BY PROFESSOR J. EDMUND WRIGHT.

THE coordinates of a unicursal curve may be expressed as rational functions of a parameter. If we assume the curve to be of order n and use non-homogeneous coordinates, we have

$$x = a(\lambda)/c(\lambda), \quad y = b(\lambda)/c(\lambda),$$

where a, b, c are polynomials of order n in the parameter λ. For the double points two values of the parameter give the same values of x and y, and the usual method for their determination consists in finding pairs of values of λ and μ that satisfy the equations

$$a(\lambda)/c(\lambda) = a(\mu)/c(\mu), \quad b(\lambda)/c(\lambda) = b(\mu)/c(\mu).$$

After elimination of μ from these equations and division ot the result by certain extraneous factors, an equation of order $(n-1)(n-2)$ in λ is obtained, and the roots of this equation combine in pairs to give the parameters of the $\frac{1}{2}(n-1)(n-2)$ double points. The process of solution however involves the solution of an equation of order $(n-1)(n-2)$.

Suppose now that a, b, c are polynomials in λ with real coefficients, *i. e.*, suppose the curve real, and write $\lambda + i\mu$ for λ. Let a be $A(\lambda, \mu^2) + i\mu A'(\lambda, \mu^2)$ and similarly for b and c. It is clear that $\lambda + i\mu$ gives for (x, y) the value

$$\begin{pmatrix} A + i\mu A' & B + i\mu B' \\ C + i\mu C', & C + i\mu C' \end{pmatrix}$$

and that $\lambda - i\mu$ gives

$$\begin{pmatrix} A - i\mu A' & B - i\mu B' \\ C - i\mu C', & C - i\mu C' \end{pmatrix}.$$

These values are the same if

$$\frac{A'C - AC'}{C^2 + \mu^2 C'^2} = 0, \quad \frac{B'C - BC'}{C^2 + \mu^2 C'^2} = 0.$$

It is at once clear that the common values of λ, μ satisfying $A'C - AC' = 0$, $B'C - BC' = 0$ give the parameters $\lambda + i\mu$, $\lambda - i\mu$ of the double points, and in addition the values of λ, μ which make $C = 0$, $C' = 0$. Also, a real pair of values of λ, μ corresponds to a real isolated double point, whilst a real crunode is given by λ real and μ purely imaginary. In either case μ^2 is real and the two above equations are in λ and μ^2; hence each real pair of values of λ, μ^2 gives a real double point and each imaginary pair an imaginary double point. For a crunode μ^2 is negative, and for an isolated point it is positive.

The number of intersections of C and C' is $n(n-1)$, whilst the number of intersections of $A'C - AC'$ and $B'C - BC'$ is apparently $(2n-1)^2$. We shall now show that this number is in reality much less.

Suppose that

$$a(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \cdots, \quad b(\lambda) = b_0\lambda^n + b_1\lambda^{n-1} + \cdots,$$
$$c(\lambda) = c_0\lambda^n + c_1\lambda^{n-1} + \cdots$$

and write

$$\lambda + i\mu = r(\cos\theta + i\sin\theta).$$

Then

$$\begin{aligned} A'C - AC' = \{&[a_0 r^n \sin n\theta + a_1 r^{n-1}\sin(n-1)\theta + \cdots] \\ &\times [c_0 r^n \cos n\theta + c_1 r^{n-1}\cos(n-1)\theta + \cdots] \\ &- [c_0 r^n \sin n\theta + c_1 r^{n-1}\sin(n-1)\theta + \cdots] \\ &\times [a_0 r^n \cos n\theta + a_1 r^{n-1}\cos(n-1)\theta + \cdots]\} \div r\sin\theta \end{aligned}$$

$$\begin{aligned} = (a_0c_1 - a_1c_0)r^{2n-2} &+ (a_0c_2 - a_2c_0)r^{2n-3}\frac{\sin 2\theta}{\sin\theta} \\ &+ \left[(a_0c_3 - a_3c_0)\frac{\sin 3\theta}{\sin\theta} + (a_1c_2 - a_2c_1)\right]r^{2n-4} \\ &+ \left[(a_0c_4 - a_4c_0)\frac{\sin 4\theta}{\sin\theta} + (a_1c_3 - a_3c_1)\frac{\sin 2\theta}{\sin\theta}\right]r^{2n-5} \\ &+ \cdots, \text{etc.,} \end{aligned}$$

$$= (a_0c_1 - a_1c_0)(\lambda^2 + \mu^2)^{n-1} + (a_0c_2 - a_2c_0)2\lambda(\lambda^2 + \mu^2)^{n-2}$$
$$+ [(a_0c_3 - a_3c_0)(3\lambda^2 - \mu^2) + (a_1c_2 - a_2c_1)(\lambda^2 + \mu^2)](\lambda^2 + \mu^2)^{n-3}$$
$$+ [(a_0c_4 - a_4c_0)(4\lambda^3 - 4\lambda\mu^2) + (a_1c_3 - a_3c_1)2\lambda(\lambda^2 + \mu^2)](\lambda^2 + \mu^2)^{n-3}$$

plus terms of order lower than $2n - 5$.

Hence $A'C - AC'$ is of order $2n - 2$ and has a multiple point of order $n - 1$ at each of the circular points at infinity. Similarly $B'C - BC'$ has multiple points at the circular points. The number of other intersections of these two curves is therefore

$$(2n - 2)^2 - 2(n - 1)^2 = 2(n - 1)^2.$$

Of these $n(n - 1)$ are accounted for, and there remain

$$2(n - 1)^2 - 2(n - 1) = (n - 1)(n - 2).$$

Now the equations contain only even powers of μ, and therefore if λ, μ be one intersection, λ, $-\mu$ is another. If μ^2 be eliminated from them, and the extraneous polynomial in λ arising from $C = 0$, $C' = 0$ be divided out, there remains an equation of order $\frac{1}{2}(n - 1)(n - 2)$ for λ. To each value of λ corresponds one double point. The corresponding value of μ^2 may in general be determined by elimination, and hence in general if λ be real the double point is real.

As an example we consider the unicursal cubic

$$x = \frac{a_0\lambda^3 + a_1\lambda^2}{c_2\lambda + c_3}, \quad y = \frac{b_1\lambda^2 + b_2\lambda}{c_2\lambda + c_3}.$$

$$A'C - AC' = a_0c_2(\lambda^2 + \mu^2)2\lambda + a_0c_3(3\lambda^2 - \mu^2)$$
$$+ a_1c_2(\lambda^2 + \mu^2) + a_1c_3 2\lambda.$$

$$B'C - BC' = b_1c_2(\lambda^2 + \mu^2) + b_1c_3 2\lambda + b_2c_3.$$

The equations for λ and μ give

(1) $$b_1c_2(\lambda^2 + \mu^2) + b_1c_3 2\lambda + b_2c_3 = 0,$$

(2) $$-4a_0b_1c_3(\lambda^2 + \mu^2) + a_1b_1c_2(\lambda^2 + \mu^2) + 2(a_1b_1c_3 - a_0b_2c_3)\lambda = 0.$$

Hence

$$\frac{2b_1c_3\lambda + b_2c_3}{b_1c_2} = \frac{2(a_1b_1c_3 - a_0b_2c_3)\lambda}{(a_1c_2 - 4a_0c_3)b_1}$$

is the equation for λ, and the value of λ from this equation, substituted in (1), gives the value of μ^2.

BRYN MAWR COLLEGE,
March, 1907.

The Ovals of the Plane Sextic Curve.

By J. Edmund Wright.

It is known that the plane sextic without double points has at most eleven separate ovals, and Hilbert has stated, though without proof, that these cannot all lie external to one another. No proof of this theorem has in fact yet been given, though Miss Ragsdale* has shown that a sextic with the maximum number of ovals, all external, cannot be obtained by the ordinary processes whereby curves with the maximum number of circuits are derived. This paper gives a proof of the theorem.

Let it be assumed that $u=0$ is a sextic with eleven external ovals all lying in the finite part of the plane, and suppose that u is chosen to be positive at infinity, then u is negative inside each oval, and positive outside. Now consider $u+c=0$ where c is a positive constant. As c increases from zero, each oval gradually shrinks up. This process of shrinking continues either until one of the ovals reduces to an isolated point, or until an oval meets itself again, thus giving an ordinary node on that oval. This last case is however impossible, for if it were possible a slight further increase in c would give a sextic with twelve ovals. A sextic, even though reducible, cannot have twelve ovals. Let then $u+c_1 \equiv v=0$ be the sextic with one isolated point and ten external ovals, no one of which contains this point. Now let A and B be two real straight lines through the point, and consider the sextic $v+k(A^2+B^2)=0$ where k is a positive constant. The particular point obtained, as k increases from zero, remains an isolated point on the sextic, and again the ovals shrink up until as before one reduces to an isolated point. Let k_1 be the value of k for this isolated point, and consider the sextic $w \equiv v+k_1(A^2+B^2)=0$. This sextic has nine external ovals, and two isolated points both lying outside all the ovals. Through

* American Journal of Mathematics (1906). *On the Arrangement of the real Branches of Plane Algebraic Curves.*

the two isolated points may be described three real conics with no other common points, say L, M, N. Then, as before, we consider the sextic

$$w + k\,(L^2 + M^2 + N^2) = 0,$$

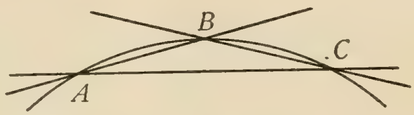
and thus obtain a sextic with three isolated points. Now suppose that we have obtained a sextic with r isolated points, and $11 - r$ external ovals, and suppose r is less than eight, we can describe through the r points three real cubics C_1, C_2, C_3, which have no other common points, and therefore if $U = 0$ be the sextic, $U + k\,(C_1^2 + C_2^2 + C_3^2) = 0$ leads to a sextic with $r + 1$ isolated points and $10 - r$ ovals, all external to each other. If $r = 8$, two real cubics may be described through the eight points, and they have one other point common. This other point can lie inside one at most of the remaining ovals, and therefore $U + k\,(C_1^2 + C_2^2) = 0$ will give a sextic with two ovals and nine external isolated points. It is clear that only one cubic can be described through these points, for if there were more there would be an infinity of them, and one of these could be made to pass through an arbitrary point on the sextic. The cubic and the sextic would then intersect in 19 points, which is impossible. Let then $U = 0$ be the sextic with nine points and two ovals, and let $C = 0$ be the cubic through the nine points. Then $U + k\,C^2 = 0$ is a sextic with the nine isolated points as double points, and C has at most one even circuit. Hence, since C can never meet the sextic again, only one at most of the two ovals of U can contain any part of C inside it,* and therefore, as before, by increasing k from zero a sextic can be obtained with ten isolated points and one oval external to the ten points. The sextic is now unicursal, and the theorem to be proved is that there can exist no unicursal sextic with ten isolated points and an external oval. The sextic thus reduces to a finite oval and ten external isolated points, each at a finite distance from the others and from the oval. Let $S = 0$ be this sextic.

Through eight of the points describe two cubics C_1, C_2 each passing through one of the other two points. Also let C_1 and C_2 be so chosen that the oval lies in the part of the plane where they are positive, and consider the sextic $S + k\,C_1 C_2 = 0$. As k increases from zero the eight points remain fixed isolated points and the two remaining points become ovals lying in the part of the plane for which $C_1 C_2$ is negative. Also the large oval shrinks up. Now suppose for any point on this oval C_1 and C_2 have the values d_1 and d_2 and $\sqrt{S_x^2 + S_y^2}$ the

* If an oval of the cubic lies inside an oval of the sextic, that oval cannot shrink up.

value p. Then if k increases from zero to a small positive value, the oval shrinks up a small amount, and the point considered moves inward a distance $k\frac{d_1 d_2}{p}$. Also suppose at the 9th point (on C_1), C_2 is δ_2, C_1 is $ax+by+\ldots.$ and S is $\alpha x^2+2\beta xy+\gamma y^2+\ldots.$, where $x=0$ $y=0$ is the point considered. The small oval is practically the ellipse $\alpha x^2+2\beta xy+\gamma y^2+k\delta_2(ax+by)=0$, so that this ellipse has dimensions proportional to k, and it moves towards the oval only when a branch of C_1 or C_2 separates it from the oval. Hence if the oval is at a finite distance from C_1 and C_2 we can thus obtain a sextic with eight isolated points and three external ovals, and the first oval is made smaller than before. We now proceed as before to shrink up the three ovals and so obtain again ten points and one oval. If the original oval should be one of the two that reduce to points, we can arrange the second process so as to stop at a smaller value of k, and thus by choosing k properly have the three ovals all points. This would give a sextic with eleven isolated points. The process must be stopped if the sextic approaches to within a small distance of one of the cubics. If however a part of the oval is at a distance $<\varepsilon$ from say C_1 it must be possible to write $S=C_1C'+\varepsilon S'$, where C' is a cubic and S' is a sextic with finite coefficients. But obviously a sextic such as that considered cannot be obtained from any two cubics and the only possibility is that C_1 consists of a straight line and a conic. If C_1 consists of a straight line and a conic, then there remains to be considered the possibility $S=PQ+\varepsilon S'$ where P is a conic and Q is a quartic, and the only possibility is that Q is a perfect square passing through the ten isolated points. This however is impossible since a conic cannot pass through more than six of the *d. p. s* of a sextic. Hence the oval is always at a finite distance away from the cubics and thus the process can be continued until the ovals reduce to eleven isolated *d. p. s*. There remains yet one possibility. The two *d. p. s* which are varied might run together. Suppose that the process is continued until the two variable points are at a very small distance apart, and then take one of these as one of the eight fixed points, allowing one of the first eight to be movable, and continue the process until these are a small distance apart if before this time the oval of the sextic cannot be reduced to a point. We can in this way keep seven points fixed and continue our process until the three points come close together. Now these points must in the limit form a point equivalent to three ordinary *d. p. s*, and it cannot be an ordinary triple point, for then a real branch of the sextic would necessarily pass through it.

It must therefore be of the type known as an oscnode, and a cubic passing through the three when they are at a small distance apart will have an ordinary point at the place. Describe the cubic through the three and six of the seven fixed points. Let the three special ones be A, B, C and consider the three



cubics through BC, CA, AB, and the other seven points. Suppose that C_1 is the cubic through the three ABC, and C_2 is, say, that through BC. Then A and the point not on C_1 turn into small ovals which may possibly approach each other. But it is easily seen that the cubics BC and CA are close to each other, and therefore if we take C_2 as the cubic CA the ovals derived from B and the point not on C_1 will be separated from one another by C_2. They therefore cannot approach very close to each other without first separating. They cannot approach across C_2. If the branch between them be the even branch of C_2 they can never approach. If it be the odd branch they can only opproach across ∞y. Hence through the whole process we may always hold six points fixed and can continue indefinitely the alternate process outlined, always arranging that the four movable points shall not run together, and we finally obtain a sextic consisting of eleven isolated points. Such a sextic would, however, be reducible, and it may easily be verified that no such reducible sextic exists. Hence the theorem is proved that *a sextic cannot consist of eleven external ovals.*

I shall show in a later paper that the only possible arrangements for the ovals of a sextic are one internal and ten external, or ten internal and one external.

BRYN MAWR, *March* 1907.

Lines of Curvature of a Surface.

By J. Edmund Wright.

It is known that a surface is intrinsically definite if the two fundamental forms ds^2 and ds^2/ρ are given. With the usual notation we represent these by

$$E\,du^2 + 2F\ du\,dv + G\ \ dv^2$$

and

$$D\,du^2 + 2D'\,du\,dv + D''\,dv^2.$$

In these two forms E, F, G may be given arbitrary values, and then D, D', D'' are subject to three relations. If the parametric lines are lines of curvature F and D' are both zero. In this case D, D'' are subject to three relations. These are not satisfied together unless a certain relation exists among the coefficients E, G of the first fundamental form. Hence if the parametric curves are lines of curvature E and G must be subject to a general condition.

If this condition is satisfied, it appears that D and D'' are in general definite and therefore the surface is intrinsically determinate.

It appears that the condition is equivalent to two differential equations of the fifth order for E and G. In this paper we determine the two equations, and apply the discussion to the particular case where $E = 1$. The remainder of the paper is concerned with a similar consideration in the case of asymptotic lines.

The equations for D, D'' are *

$$\frac{D\,D''}{\sqrt{E\,G}} + \frac{\partial}{\partial u}\left(\frac{1}{\sqrt{E}}\,\frac{\partial\sqrt{G}}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{1}{\sqrt{G}}\,\frac{\partial\sqrt{E}}{\partial v}\right) = 0 \tag{1}$$

$$\frac{\partial}{\partial v}\left(\frac{D}{\sqrt{E}}\right) = \frac{D''}{G}\,\frac{\partial\sqrt{E}}{\partial v}, \tag{2}$$

$$\frac{\partial}{\partial u}\left(\frac{D''}{\sqrt{G}}\right) = \frac{D}{E}\,\frac{\partial\sqrt{G}}{\partial u}. \tag{3}$$

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* See Bianchi–Lukat, p. 94.

We write $D/\sqrt{E}=X$, $D''/\sqrt{G}=Y$, $E=A^2$, $G=C^2$, $p=\frac{1}{A}\frac{\partial C}{\partial u}$ $q=\frac{1}{C}\frac{\partial A}{\partial v}$, and use suffixes to denote differentiations with respect to u and v. The equations now become

$$XY+p_1+q_2=0, \qquad X_2=qY, \qquad Y_1=pX. \qquad (4)\ (5)\ (6)$$

If we write $p_1+q_2=-P$ we have $\frac{\partial}{\partial v}X^2=2qP$, $\frac{\partial}{\partial u}Y^2=2pP$. Hence $X^2=2\int qP\,dv$, $Y^2=2\int pP\,du$, where an arbitrary function enters through each of the integrals, and

$$P^2=4\int pP\,du\int qP\,dv. \qquad (7)$$

It is clear that, if this relation is satisfied, the equations are compatible, and therefore a surface exists with the given element of length, for which the parametric lines are lines of curvature.

We now write $P^2=a$, $2qP=b$, $2pP=c$, $X^2=\xi$, $Y^2=\eta$ and our equations become $\xi\eta=a$, $\xi_2=b$, $\eta_1=c$.

Consider first the case in which a, b, c, are all different from zero; ξ must satisfy the two equations $\frac{\partial}{\partial u}\left(\frac{a}{\xi}\right)=c$, $\frac{\partial\xi}{\partial v}=b$. Hence

$$\frac{a_1}{c}\xi-\xi^2-\frac{a}{c}\xi_1=0. \qquad (8)$$

If we differentiate this with respect to v we get

$$(a_1b-ab_1)\,c+(a_{12}c-a_1c_2-2bc^2)\,\xi+(ac_2-a_2c)\,\xi_1=0. \qquad (9)$$

Now if (a_1b-ab_1) and (ac_2-a_2c) are both zero, $a_{12}c-a_1c_2-2bc^2=0$ since $\xi\neq 0$.

If ac_2-a_2c only is zero, (9) determines ξ uniquely and then η is uniquely determined. In any case (8) and (9) determine ξ as the root of a quadratic equation, unless (9) is nugatory, and ξ_1 is determinate. We assume for the present that $ac_2-a_2c\neq 0$. These equations give $\xi=\alpha$, say, and $\xi_1=\beta$. Also $\xi_2=b$. Hence we have two necessary and sufficient conditions for coexistence, namely $\alpha_2=b$, $\alpha_1=\beta$. These two conditions are of the third order in a, b, c, or of the fifth order in A and C. It is obvious that if they are satisfied, values of ξ and η can be determined to satisfy our system of equations, and further, there are only two possible sets of solutions.

We now show that in general there is only one such set; in fact, divide (9) by $ac_2 - a_2 c$ and differentiate again with respect to v. We get

$$\frac{(a_1 b - ab_1)(a_{22} c - ac_{22})c}{(ac_2 - a_2 c)^2} + \frac{2a_{12} bc - acb_{12} - 2b^2 c^2 + c(a_1 b_2 - 2a_2 b_1)}{ac_2 - a_2 c} + \frac{\partial}{\partial v}\left(\frac{a_{12} c - a_1 c_2 - 2bc^2}{ac_2 - a_2 c}\right)\xi = 0. \qquad (10)$$

If the coefficient of ξ in this equation is not zero, ξ is determined uniquely, and hence there is only one set. If this coefficient is zero we form the corresponding set of equations with η instead of ξ, and again we see that ξ and η are uniquely determinate unless

$$\frac{\partial}{\partial u}\left(\frac{a_{12} b - a_2 b_1 - 2b^2 c}{ab_1 - a_1 b}\right) = 0.$$

Hence unless both the expressions named are zero the surface having the given linear element, and the parametric lines as lines of curvature, is intrinsically determinate. Looking away from these particular cases we see that the two general equations of condition may be determined by eliminating ξ, ξ_1, from (8), (9), (10), and η, η_2, from the corresponding equations in η. The two equations thus obtained are obviously of the fifth order in A, C, and they are clearly independent, for fifth derivatives enter only through a_{122} in the first, and through a_{112} in the second.

The only case where ξ and η are not determinate arises if (9) is nugatory. We thus have the results:

(i) If the linear element of a surface is given by $ds^2 = A^2 du^2 + C^2 dv^2$, and if the parametric curves are lines of curvature, the surface in general is intrinsically uniquely determinate, and A and C must satisfy two equations involving their fifth derivatives. These two equations of condition are necessary and sufficient.

(ii) In certain cases there may be two surfaces satisfying the given conditions.

(iii) In certain cases there may be an infinite number of such surfaces.

We shall now enquire more closely into the surfaces of classes (ii) and (iii). The equations arising from the identical vanishing of (10) and its correspondent are rather cumbrous, so we consider the question again *ab initio*. We assume, in fact, that the equations $\frac{\partial}{\partial u}\left(\frac{a}{\xi}\right) = c$ and $\frac{\partial \xi}{\partial v} = b$ have two different solutions,

ξ and ξ'. This is equivalent to the case arising from the identical vanishing of (10) and the corresponding equation, with the exception that we must treat independently the case when the quadratic for ξ is a perfect square. We deduce at once that $\xi - \xi' = U$, $a\frac{\xi - \xi'}{\xi\xi'} = V$, when U and V are functions of u and v alone in that order. We still assume that a, b, c are different from zero, and therefore, since $\xi \neq \xi'$, U and V are both different from zero. We now take as new independent variables $\int \sqrt{U}\, du$ and $\int \sqrt{V}\, dv$, and we have $\xi - \xi' = 1$, $\xi\xi' = a$. Hence from the quadratic for ξ we get

$$\frac{a_1 b - ab_1}{a_2 c - ac_2} = 1. \tag{11}$$

$$\left\{\frac{a_{12} c - a_1 c_2 - 2bc^2}{ac_2 - a_2 c} + \frac{a_1}{a}\right\} \frac{a}{c} = \xi + \xi'.$$

Also if we substitute in turn ξ and ξ' in (9) and substract, we deduce

$$a_{12} c - a_1 c_2 - 2bc^2 = 0. \tag{12}$$

Hence $\xi + \xi' = \frac{a_1}{c}$. By substituting the values of the various coefficients in (9) we obtain $\xi_1 = c$, and therefore $c_2 = b_1$. Hence $c = \bar{\omega}_1$, $b = \bar{\omega}_2$, where $\bar{\omega}$ is a function of u and v. Also (11) becomes $a_1 \bar{\omega}_2 - a_2 \bar{\omega}_1 = 0$, and therefore $a = f(\bar{\omega})$. With these values the quadratic for ξ is $f'(\bar{\omega})\, \xi - \xi^2 - f(\bar{\omega}) = 0$, and (12) becomes $f'' - 2 = 0$.

$f(\bar{\omega})$ must therefore satisfy the two equations $f'' - 2 = 0$ and $1 = (\xi - \xi')^2 = f'^2 - 4f$. The first is seen to be a consequence of the second, and $f = (\bar{\omega} + \text{const.})^2 - \frac{1}{4}$. Our equations are now all satisfied and if we incorporate the constant into $\bar{\omega}$ we have

$$a = \bar{\omega}^2 - \tfrac{1}{4}, \quad b = \bar{\omega}_2, \quad c = \bar{\omega}_1;$$

$$\xi = \bar{\omega} + \tfrac{1}{2}, \quad \xi' = \bar{\omega} - \tfrac{1}{2}, \quad \eta = \bar{\omega} - \tfrac{1}{2}, \quad \eta' = \bar{\omega} + \tfrac{1}{2}.$$

We still have the relations

$$a = (p_1 + q_2)^2, \quad b = -2q(p_1 + q_2), \quad c = -2p(p_1 + q_2).$$

These give $p_1 + q_2 = \sqrt{\bar{\omega}^2 - \frac{1}{4}}$

$$p = -\frac{\bar{\omega}_1}{2\sqrt{\bar{\omega}^2 - \frac{1}{4}}}, \; q = \frac{-\bar{\omega}_2}{2\sqrt{\bar{\omega}^2 - \frac{1}{4}}}$$

Hence

$$\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{\bar{\omega}^2 - \frac{1}{4}}}\frac{\partial \bar{\omega}}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{1}{\sqrt{\bar{\omega}^2 - \frac{1}{4}}}\frac{\partial \bar{\omega}}{\partial v}\right) + 2\sqrt{\bar{\omega}^2 - \tfrac{1}{4}} = 0.$$

We write $\omega = \frac{1}{2}\cosh\omega$ and this becomes

$$\frac{\partial^2\omega}{\partial u^2} + \frac{\partial^2\omega}{\partial v^2} + \sinh\omega = 0. \tag{13}$$

In this investigation no distinction has been made between real and imaginary quantities, but the connection between surfaces with the same spherical representation as spherical or pseudospherical surfaces is immediate.*

We now consider the case when the quadratic for ξ has equal roots, and the equations of type (10) are nugatory. We have $(a_{12}c - a_1c_2 - 2bc^2)/(ac_2 - a_2c)$ equal to a function of u only. Let this function be $\frac{1}{U}\frac{dU}{du}$, and choose $\int\sqrt{U}\,du$ as the variable u. Similarly choose the new variable v. We now have

$$a_{12}c - a_1c_2 - 2bc^2 = 0,$$
$$a_{12}b - a_2b_1 - 2b^2c = 0,$$

and there is still a constant factor at our disposal in each of the variables u, v. From these two equation we deduce

$$\frac{a_1b}{a_2c} = \frac{b_1}{c_2},$$

and therefore each of these fractions is equal to

$$\frac{a_1b - ab_1}{a_2c - ac_2}.$$

If we substitute these values in (9) we get $\xi_1 = \frac{a_1}{a_2}b$, and therefore $\xi_1a_2 = \xi_2a_1$. ξ is therefore a function of a only, say $f(a)$. The quadratic for ξ is

$$\frac{a_1}{c}\xi - \xi^2 = \frac{aa_1b}{a_2c},$$

and since this is a perfect square we have

$$a_1a_2 = 4abc, \quad \xi = \tfrac{1}{2}\frac{a_1}{c}.$$

Now

$$\xi = f(a),\ \xi_2 = b,\ \frac{\partial}{\partial v}\left(\frac{a}{\xi}\right) = c,$$

and therefore

$$f'(a)a_1 = b, \left(\frac{1}{f} - \frac{af'}{f^2}\right)a_2 = c.$$

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* See Bianchi-Lukat, p. 472. The relation between the two surfaces corresponds to Hazzidaki's transformation. Eisenhart, Amer. Journal, Vol. 27, pp. 113–172, and Amer. Journal, Vol. 28, pp. 47–70.

Hence

$$\left(\frac{f'}{f} - a\frac{f'^2}{f^2}\right)a_1 a_2 = bc,$$

or

$$4a\left(\frac{f'}{f} - a\frac{f'^2}{f^2}\right) = 1.$$

We thus have $2af' - f = 0$, or $f = k\sqrt{a}$, where k is a constant. By choosing appropriately the constant factors at our disposal in u and v, we make $k = 1$, and we now have $\xi = \sqrt{a} = \eta$.

Further $\xi_2 = b$, $\eta_1 = c$, and hence

$$b = \frac{\partial\sqrt{a}}{\partial v}. \quad c = \frac{\partial\sqrt{a}}{\partial u}.$$

If we put $a = \bar{\omega}^2$ we have

$$a = \bar{\omega}^2,\ b = \bar{\omega}_2,\ c = \bar{\omega}_1$$
$$\xi = \bar{\omega},\ \eta = \bar{\omega}.$$

Hence $p_1 + q_2 = \bar{\omega}$, $p = -\frac{1}{2}\frac{\bar{\omega}_1}{\bar{\omega}}$, $q = -\frac{1}{2}\frac{\bar{\omega}_2}{\bar{\omega}}$, and the equation for ω is

$$\frac{\partial}{\partial u}\left(\frac{1}{\bar{\omega}}\frac{\partial\bar{\omega}}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{1}{\bar{\omega}}\frac{\partial\bar{\omega}}{\partial v}\right) + 2\bar{\omega} = 0,$$

or if we put $-2\bar{\omega} = e^{\omega}$

$$\frac{\partial^2\omega}{\partial u^2} + \frac{\partial^2\omega}{\partial v^2} = e^{\omega}. \tag{14}$$

The spherical representation for this class of surfaces is

$$d\sigma^2 = \bar{\omega}(du^2 + dv^2).*$$

We now consider (iii), for which the quadratics for ξ and η are nugatory, and a, b, c are none of them zero. In the first place suppose $ac_2 - a_2 c = 0$, and that the remaining coefficients in (9) do not vanish. We have $\frac{c}{a} = U$, and by taking $\int U\,du$ as a new variable u, we have $c = a$.

The equations for ξ become

$$\frac{a_1}{a}\xi - \xi^2 = \xi_1.$$

$$(a_{12}a - a_1 a_2 - 2a^2 b)\,\xi = a(ab_1 - a_1 b),$$

and we have, as before, a unique value of ξ, and two equations of condition

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*In this connection see Bianchi p. 135 sqq.

between a and b. There is a further limitation since $c=a$. This involves $p_1+q_2+2p=0$, and therefore

$$\frac{\partial}{\partial u}\left(e^{2u}p\right)+\frac{\partial}{\partial v}\left(e^{2u}q\right)=0;$$

hence if ϕ is a certain function of u and v we have

$$e^{2u}p=\frac{\partial\phi}{\partial v},\quad e^{2u}q=-\frac{\partial\phi}{\partial u},$$

and

$$a=\quad 4e^{-4u}\left(\frac{\partial\phi}{\partial v}\right)^2$$

$$b=-\ 4e^{-4u}\frac{\partial\phi}{\partial u}\ \frac{\partial\phi}{\partial v}.$$

We now suppose that the quadratic for ξ is nugatory. We have $a=c$, and $a_1b-ab_1=0$. Hence by suitable choice of the variable v we may make $a=b$.

In addition, (9) shows that we must have $aa_{12}-a_1a_2-2a^3=0$. If this condition is satisfied the equations $\xi\eta=a$, $\xi_2=a$, $\eta_1=a$ are compatible, but we must also have

$$-(p_1+q_2)=2p=2q=\sqrt{a}.$$

Hence a must also satisfy

$$\frac{\partial}{\partial u}(\sqrt{a})+\frac{\partial}{\partial v}(\sqrt{a})+2\sqrt{a}=0.$$

From this equation $\sqrt{a}=e^{-2u}F(u-v)$, where F is an arbitrary function of its argument. But this value of a cannot satisfy the first condition for any value of F. Hence there are no surfaces of the kind sought.

We now consider the possibility hitherto neglected, namely that one of the quantities a, b, c is zero. Going back to the equations (4), (5), (6), we see that the condition requires either p or q to be zero. We assume $q=0$ and we have

$$XY+p_1=0,\quad X_2=0,\quad Y_1=pX,$$

together with $\dfrac{\partial A}{\partial v}=0$. Hence A is a function of u only, and we may by suitable choice of u, make it unity. We have now

$$ds^2=du^2+C^2\,dv^2,$$

and $p=C_1$. The equations are readily solved; there are two possibilities accord-

ing as X is or is not zero. In the first case the surface is of zero curvature, and $ds^2 = du^2 + (au + b)^2 dv^2$, where a and b are functions of v only.

In the second case let $X = x'$, where x' is the first derivative of a function x of u only. We now have $X = x'$, $C_1 = V_1 \cos x + V_2 \sin x$, $Y = V_1 \sin x - V^2 \cos x$, where V_1 and V_2 are functions of v only.

The conditions C must satisfy, in order that it may be thrown into the given form, are

$$\begin{vmatrix} C_1 & C_{11} & C_{111} \\ C_{12} & C_{112} & C_{1112} \\ C_{122} & C_{1122} & C_{11122} \end{vmatrix} = 0,$$

and

$$\frac{\partial}{\partial u}\left[\frac{C_{11}\ C_{1112} - C_{112}\ C_{111}}{(C_1\ C_{112} - C_{11}\ C_{12})^3}\right] = 0.$$

The spherical representation is given by

$$d\sigma^2 = x'^2\, du^2 + (V_1 \sin x - V_2 \cos x)^2\, dv^2.$$

It readily follows that the surface is that swept out by a curve fixed in a given plane when the plane rolls on a developable surface.*

We now consider the case of a surface with linear element

$$ds^2 = E\, du^2 + 2F\, du\, dv + G\, dv^2,$$

for which the parametric lines are asymptotic. We have $D = D'' = 0$, and †

$$\frac{\partial}{\partial u}\frac{D'}{\sqrt{EG - F^2}} = -2\begin{Bmatrix} 1\ .\ 2 \\ 2 \end{Bmatrix}\frac{D'}{\sqrt{EG - F^2}}$$

$$\frac{\partial}{\partial v}\frac{D'}{\sqrt{EG - F^2}} = -2\begin{Bmatrix} 1\quad 2 \\ 1 \end{Bmatrix}\frac{D'}{\sqrt{EG - F^2}}.$$

Also if K is the total curvature

$$\frac{D'^2}{EG - F^2} = -K.$$

Hence E, F, G must be subject to the two conditions

$$\frac{\partial}{\partial u}\log K = -4\begin{Bmatrix} 1\quad 2 \\ 2 \end{Bmatrix}$$

$$\frac{\partial}{\partial v}\log K = -4\begin{Bmatrix} 1\ .\ 2 \\ 1 \end{Bmatrix}.$$

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* See also Bianchi–Lukat, p. 166.

† Bianchi–Lukat, p. 92.

These conditions are seen to be equivalent to two, one of the second and the other of the third in derivatives of E, F, G.

As a particular example, consider the case in which the asymptotic lines cut at right angles. The equations of condition give immediately

$$KE^2 = U, \qquad KG^2 = V,$$

where U is a function of u only, V is a function of V only. By appropriate choice of u and v we make $KE^2 = KG^2 = 1$.

We also have the additional equation

$$\frac{\partial^2 V}{\partial u^2} + \frac{\partial^2 V}{\partial v^2} = e^{\frac{1}{2}V},$$

where $\qquad K = e^V.$

Hence $$\sqrt{K} = \frac{4f'(u+iv)\,\phi'(u-iv)}{[f(u+iv)+\phi(u-iv)]^2}.$$

Also $H = 0$, and the spherical representation is given by $d\sigma^2 = -\sqrt{K}(du^2 + dv^2)$. We note that since $ds^2 = \dfrac{1}{\sqrt{K}}(du^2 + dv^2)$ the spherical representation is conformal, and the general properties of minima surfaces may be readily obtained from this point of view.

Suppose next that only one set of parametric curves, say that for which $v =$ constant, is asymptotic. In this case $D = 0$, and

$$\frac{D'}{\sqrt{EG - F^2}} = \sqrt{-K}.$$

In addition we must have

$$\frac{\partial}{\partial u}\sqrt{-K} = -2\begin{Bmatrix}1 & 2\\ & 2\end{Bmatrix}\sqrt{-K} + \begin{Bmatrix}1 & 1\\ & 2\end{Bmatrix}\frac{D''}{\sqrt{EG-F^2}},$$

$$\frac{\partial}{\partial u}\left(\frac{D''}{\sqrt{EG-F^2}}\right) - \frac{\partial}{\partial v}\sqrt{-K} =$$

$$2\begin{Bmatrix}1 & 2\\ & 1\end{Bmatrix}\sqrt{-K} + \begin{Bmatrix}1 \,.\, 1\\ 1\end{Bmatrix}\frac{D''}{\sqrt{EG-F^2}}.$$

There is thus one equation of condition, of the fourth order, among the quantities E, F, G, and their derivatives, and this is given by substituting the value of D'' given by the first in the second of the above two equations. If this

condition is satisfied D' and D'' are determinate and therefore the surface also is intrinsically determinate. As a particular case we readily deduce that if $E=1$, $F=0$, G must equal au^2+bu+c, where a, b, c are three arbitrary functions of v only. This is of course the case of skew surfaces.

The general question as to what conditions are necessary in order that the parametric curves may be conjugates for some surface with a given linear element may also be treated in the same manner as that connected with the lines of curvature. We have to consider the possibility of coexistence of three equations

$$\xi\eta = a, \quad \xi_2 = \alpha\xi + \beta, \quad \eta_1 = \gamma\eta + \delta,$$

when a, α, β, γ, δ are known functions of u, v, we find that there must exist two relations of the fifth order among E, F, G and their derivatives.

BRYN MAWR, *April* 1906.

Corresponding dynamical Systems.

(*By* J. Edmund Wright, *à Bryn Mawr, Penn.*)

Let

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_r}\right)-\frac{\partial T}{\partial x_r}=X_r, \qquad (r=1,\ 2,\ldots,\ n) \tag{1}$$

where

$$2\,T=\sum_{r,s=1}^{n} a_{rs}\,\dot{x}_r\,\dot{x}_s,$$

be the Lagrange equations of a holonomic dynamical system, and suppose the quantities a_{rs}, X_r not to involve the velocities $\dot{x}$, but to be simply functions of position of the coordinates x.

The complete solution of the equations (1) will give expressions for the variables x as functions of t and $2\,n$ arbitrary constants. If we suppose the arbitrary constants given, we have the variables expressed as functions of one parameter, and thus the particular solution considered may be regarded as determining a curve is space of n dimensions. The movement of the system, corresponding to this curve, is completely determined when we know at what speed the curve is described. Such a curve is called a trajectory.

Again, let

$$\frac{d}{dt_1}\left(\frac{\partial \mathfrak{T}}{\partial x'_r}\right)-\frac{\partial \mathfrak{T}}{\partial x_r}=Y_r, \qquad (r=1,\ 2,\ldots,\ n) \tag{2}$$

where

$$2\,\mathfrak{T}=\sum_{r,s=1}^{n} c_{rs}\,x'_r\,x'_s,$$

and accents are used to denote derivatives with respect to t_1, be another such system.

Any solution of (1) is of the type $x_i=f_i(t)$, $(i=1,\ 2,\ldots,\ n)$, and any solution of (2) is of the type $x_i=\varphi_i(t_1)$, $(i=1,\ 2,\ldots,\ n)$, when f_i and φ_i are certain functions of t or t_1.

It may be possible to choose t_1 as such a function of t that $f_i(t) = \varphi_i(t_1)$ for all values of i, and in this case the same curve will be a trajectory of both systems. The motions of the two systems will not in general correspond even though they are represented by the same trajectory, for the velocities along the trajectory will in general be different for the two systems. They will be the same only if $t = t_1 + \text{const}$.

Painlévé (*) has proposed the question: What are the conditions in order that two dynamical systems may have the same trajectories? He has not given the solution of the problem, but has obtained some general conditions to which the two systems must be subject, for example, they must possess quadratic integrals of geodesics, etc.

The simpler problem in which all the forces X are zero has been completely solved by Levi-Civita (**) as an application of the methods of the *Absolute Differential Calculus* (***).

The present paper contains a discussion of the whole problem, and the general equations of condition are obtained in a simpler manner than has been done by the authors quoted. The Absolute Calculus and the theory of congruences of Ricci (****) are used to obtain certain particular corresponding systems which have not yet been determined.

§ 1.

We first recall the notation and elementary processes of the Absolute Calculus. Let there be given a quadratic differential form

$$\sum_{r,s=1}^{n} a_{rs}\, d x_r\, d x_s ,$$

where the $a's$ are functions of the $x's$, and suppose that when a general transformation is performed on the $x's$ this becomes

$$\sum_{r,s=1}^{n} a'_{rs}\, d x'_r\, d x'_s .$$

(*) *Liouville's Journal*, Ser. 4, Vol. 10 (1894), pp. 5-92.

(**) *Annali di Mat.*, Vol. 24 (1896), pp. 255-300.

(***) Ricci and Levi-Civita, *Math. Ann.*, Vol. 54 (1901), pp. 125-201.

(****) *Loc. cit.*, pp. 145 sqq.

We then have

$$a'_{rs} = \sum_{p,q=1}^{n} a_{pq} \frac{\partial x_p}{\partial x'_r} \frac{\partial x_q}{\partial x'_s} \qquad (r,\ s = 1,\ 2, \ldots,\ n).$$

Suppose that a system of quantities $X_{r_1 r_2 \ldots r_m}$ is such that when we effect the transformation on the $x's$ the elements of the transformed system X' are given by the equations

$$X'_{r_1 r_2 \ldots r_m} = \sum_{s_1 s_2 \ldots s_m = 1}^{n} X_{s_1 s_2 \ldots s_m} \frac{\partial x_{s_1}}{\partial x'_{r_1}} \frac{\partial x_{s_2}}{\partial x'_{r_2}} \cdots \frac{\partial x_{s_m}}{\partial x'_{r_m}},$$

then the system X is said to be *covariant* of the m^{th} order. A system $X^{(r_1 r_2 \ldots r_m)}$ for which the equations of transformation are

$$\left\{ X^{(r_1 r_2 \ldots r_m)} \right\}' = \sum_{s_1 s_2 \ldots s_m = 1}^{n} X^{(s_1 s_2 \ldots s_m)} \frac{\partial x'_{r_1}}{\partial x_{s_1}} \frac{\partial x'_{r_2}}{\partial x_{s_2}} \cdots \frac{\partial x'_{r_m}}{\partial x_{s_m}}$$

is said to be *contravariant* of the m^{th} order.

The coefficients a_{rs} are thus seen to be a covariant system of the second order. Let a denote the determinant whose element in the r^{th} row and s^{th} column is a_{rs}, and let A_{rs} be the cofactor of this element in a. Then if $a^{(rs)} = \dfrac{A_{rs}}{a}$ it is easy to see that the quantities $a^{(rs)}$ are a contravariant system of the second order, and further that

$$\sum_{t=1}^{n} a^{(rt)} a_{st} = 0 \quad (r \neq s)$$

and

$$\sum_{t=1}^{n} a^{(rt)} a_{rt} = 1.$$

The notation $[r\,s,\ t]$, $\{r\,s,\ t\}$, is used for Christoffel's three index symbols of the first and second kinds, so that

$$[r\,s,\ t] = \frac{1}{2}\left(\frac{\partial}{\partial x_s} a_{rt} + \frac{\partial}{\partial x_r} a_{st} - \frac{\partial}{\partial x_t} a_{rs}\right)$$

$$\{r\,s,\ t\} = \sum_{k=1}^{n} [r\,s,\ k]\, a^{(kt)}.$$

If we are given a covariant system $X_{r_1 r_2 \ldots r_m}$ of order m, we can obtain from it a system of order $m+1$ by a process known as *covariant differen-*

tiation. The equation for an element of the derived system is

$$X_{r_1 r_2 \dots r_{m+1}} = \frac{\partial}{\partial x_{r_{m+1}}} (X_{r_1 \dots r_m}) - \sum_{l=1}^{m} \sum_{q=1}^{n} \{r_l\, r_{m+1},\ q\}\ X_{r_1 \dots r_{l-1} q r_{l+1} \dots r_m}.$$

Thus, for example, if f is an invariant

$$f_r = \frac{\partial f}{\partial x_r}$$

gives its first derived system. Its second and third derived systems are

$$f_{rs} = \frac{\partial f_r}{\partial x_s} - \sum_{q=1}^{n} \{r s,\ q\} f_q$$

$$f_{rst} = \frac{\partial f_{rs}}{\partial x_t} - \sum_{q=1}^{n} \{r t,\ q\} f_{qs} - \sum_{q=1}^{n} \{s t,\ q\} f_{rq}.$$

Note that the suffixes of a derived system are not in general interchangeable, $e.g.\ f_{rst} \neq f_{rts}$. For example, if X_r be a covariant system of order unity

$$X_{rst} - X_{rts} = \sum_{pq} a^{(pq)} (s\, t\, r\, p)\, X_q$$

where the quantities $(s\,t\,r\,p)$ are certain functions of the $a's$ and their derivatives, known as Riemann symbols, or Christoffel four index symbols. An important result in this connection is that if $X_{rs} = X_{sr}$, and X_{rs} is derived from a system X_r, then X_r is the derived system of an invariant X.

If we have a covariant system, for example X_{rst}, the system

$$X^{(rst)} \equiv \sum_{p,q,k=1}^{n} a^{(rp)}\, a^{(sq)}\, a^{(tk)}\, X_{pqk}$$

is contravariant of the same order. Also

$$X_{rst} = \sum_{p,q,k=1}^{n} a_{rp}\, a_{sq}\, a_{tk}\, X^{(pqk)}$$

and hence the two systems X are said to be reciprocal with respect to the fundamental form.

Consider a system of equations such as

$$\frac{d\,x_1}{\lambda^{(1)}} = \frac{d\,x_2}{\lambda^{(2)}} = \dots = \frac{d\,x_n}{\lambda^{(n)}}$$

and let $\sum\limits_{r,s=1}^{n} a_{rs}\,\lambda^{(r)}\,\lambda^{(s)} = 1$. Then the system λ is contravariant of the first order, and $\sum\limits_{r=1}^{n} \lambda^{(r)}\,\lambda_r = 1$. Also if the fundamental form is regarded as $d\,s^2$ for a certain manifold, the above equations define a congruence of curves in the manifold. Let i and j be used to denote two such congruences, then if the two congruences cut at right angles it may easily be proved that

$$\sum_r \lambda_{i|r}\,\lambda_j^{(r)} = \sum_r \lambda_i^{(r)}\,\lambda_{j|r} = 0.$$

Corresponding to any manifold of order n there may be determined in an infinite number of ways a system of n mutually orthogonal congruences. Such a set of congruences is called an orthogonal ennuple. If the congruences of the ennuple are denoted by suffixes 1, 2,..., n, it may easily be proved that any covariant system may be expressed in terms of the quantities $\lambda_{i|r}$, $(i,\ r = 1,\ 2,\ldots,\ n)$, and invariants. For example, if X_{vst} is a covariant system of the third order and

$$c_{\alpha\beta\gamma} = \sum_{r,s,t=1}^{n} X_{rst}\,\lambda_\alpha^{(r)}\,\lambda_\beta^{(s)}\,\lambda_\gamma^{(t)},$$

then $c_{\alpha\beta\gamma}$ is an invariant, and

$$X_{rst} = \sum_{\alpha,\beta,\gamma=1}^{n} c_{\alpha\beta\gamma}\,\lambda_{\alpha|r}\,\lambda_{\beta|s}\,\lambda_{\gamma|t}\,.$$

In particular for the first derived system of the system $\lambda_{h|r}$ we have

$$\lambda_{h|rs} = \sum_{i,j=1}^{n} \gamma_{hij}\,\lambda_{i|r}\,\lambda_{j|s}\,.$$

The invariants γ_{hij} are called the coefficients of rotation of the ennuple. Among these invariants there exist the relations $\gamma_{hij} + \gamma_{ihj} = 0$, and in particular $\gamma_{hhj} = 0$.

The necessary and sufficient conditions that the congruence n be composed of the normal trajectories of a family of surfaces is that $\gamma_{nhk} = \gamma_{nkh}$. In particular, if all the congruences are normal, it readily follows that all the $\gamma' s$ with three different suffixes are zero.

Let there be a covariant system of the second order c_{rs}, and consider the determinant $\Delta \equiv |\, c_{rs} - \rho\, a_{rs}\,|$. It is of the n^{th} degree in ρ, and if the roots

of $\Delta = 0$ are $\rho_1, \rho_2, \ldots, \rho_n$, it may be proved that an orthogonal ennuple can be determined such that

$$a_{rs} = \sum_{h=1}^{n} \lambda_{h|r} \lambda_{h|s}, \quad c_{rs} = \sum_{h=1}^{n} \rho_h \lambda_{h|r} \lambda_{h|s}.$$

The functions ρ_h are of course invariants (*).

§ 2.

We need in the course of our work the following lemmas.

I) Suppose $Q_1, Q_2, \ldots, Q_n$ to be homogeneous quadratics in n variables, $L_1, L_2, \ldots, L_n$ linear in the same variables, and $Y_1, Y_2, \ldots, Y_n$ constants, not all zero. If the determinant of the coefficients of the $L's$ do not vanish and if all the three rowed determinants of the matrix

$$\left\| \begin{matrix} Q_1, & Q_2, \ldots, & Q_n \\ L_1, & L_2, \ldots, & L_n \\ Y_1, & Y_2, \ldots, & Y_n \end{matrix} \right\|$$

vanish identically then

$$Q_r = b\, L_r + R\, Y_r \qquad (r = 1, 2, \ldots, n)$$

where b is linear and R quadratic in the variables.

For consider the identity

$$\begin{vmatrix} Q_1 & Q_2 & Q_3 \\ L_1 & L_2 & L_3 \\ Y_1 & Y_2 & Y_3 \end{vmatrix} = 0$$

and assume that $Y_1 \neq 0$. This gives

$$(Y_1 Q_2 - Y_2 Q_1)(Y_1 L_3 - Y_3 L_1) \equiv (Y_1 Q_3 - Y_3 Q_1)(Y_1 L_2 - Y_2 L_1).$$

(*) For the detailed exposition of the results of the above section the reader should see the paper of Ricci and Levi-Civita already quoted, or Wright, *Cambridge Tract.* (N.º 9), *Invariants of Quadratic Differential Forms* (1908).

Now no linear relation can exist among L_1, $L_2, \ldots,$ L_3, and therefore the above identity shows that the linear quantity $Y_1 L_3 - Y_3 L_1$, must be a factor of the quadratic $Y_1 Q_3 - Y_3 Q_1$.

We therefore have

$$Y_1 Q_2 - Y_2 Q_1 = b\,(Y_1 L_2 - Y_2 L_1)$$
$$Y_1 Q_3 - Y_3 Q_1 = b\,(Y_1 L_3 - Y_3 L_1)$$

where b is homogeneous and linear is the variables.

If we put $Q_1 = b\,L_1 + R\,Y$, R is a homogeneous quadratic and the above equations give

$$Q_2 = b\,L_2 + R\,Y_2$$
$$Q_3 = b\,L_3 + R\,Y_3$$

and hence in general we have $Q_r = b\,L_r + R\,Y_r$, which proves the theorem.

II) With the same notation as above, suppose all the $Y's$ zero. Then if all the two rowed determinants of the matrix

$$\left\| \begin{matrix} Q_1, & Q_2, \ldots, & Q_n \\ L_1, & L_2, \ldots, & L_n \end{matrix} \right\|$$

are identically zero we must have

$$Q_r = b\,L_r \qquad (r = 1, 2, \ldots, n)$$

where b is linear and homogeneous.

This follows as before, since, for example,

$$Q_1 L_2 - Q_2 L_1 = 0,$$

and L_2 cannot divide L_1.

§ 3

Now let

$$\frac{d}{d\,t}\left(\frac{\partial T}{\partial \dot{x}_r}\right) - \frac{\partial T}{\partial x_r} = X_r\,, \qquad 2\,T \equiv \sum_{r,s=1}^{n} a_{rs}\,\dot{x}_r\,\dot{x}_s\,, \tag{1}$$

$$\frac{d}{d\,t_1}\left(\frac{\partial \mathfrak{T}}{\partial x'_r}\right) - \frac{\partial \mathfrak{T}}{\partial x_r} = Y_r\,, \qquad 2\,\mathfrak{T} \equiv \sum_{r,s=1}^{n} c_{rs}\,x'_r\,x'_s\,, \tag{2}$$

$$(r = 1, 2, \ldots, n)$$

be two corresponding systems. Tahe $\Sigma a_{rs}\, d x_r\, d x_s$ for the fundamental form, and along a selected trajectory let $t = \theta(t_1)$. If the equations (1) are solved for the second derivatives of the $x's$ with respect to t, they give

$$\ddot{x}_r = X^{(r)} - \sum_{p,q=1}^{n} \{p\, q,\, r\}\, \dot{x}_p\, \dot{x}_q \qquad (r = 1, 2, \ldots, n) \tag{3}$$

Along the selected trajectory we have

$$x'_r = \dot{x}_r\, \theta', \qquad x''_r = \ddot{x}_r\, \theta'^2 + \dot{x}_r\, \theta''$$

where $\theta' \equiv \frac{\partial \theta}{d t_1}$, $\theta'' = \frac{d^2 \theta}{d t_1^2}$.

If the values of second derivatives are substituted in the typical equation (2) it becomes

$$\sum_{s=1}^{n} c_{rs}\, \dot{x}_s\, \theta'' + \sum_{s=1}^{n} c_{rs}\, X^{(s)}\, \theta'^2 + \sum_{p,q=1}^{n} k_{pqr}\, \dot{x}_p\, \dot{x}_q\, \theta'^2 = Y_r \tag{4}$$

where k_{pqr} is written for $\frac{1}{2} \left\{ c_{prq} + c_{qrp} - c_{pqr} \right\}$.

The equations (4) are n in number, but they cannot be equivalent to more than two. Suppose in fact that three among them were independent, then we could eliminate from these three the magntudes θ'', θ'^2, and we should thus obtain a relation among the magnitudes x, $\dot{x}$, free from any arbitrary constant, which would hold for all trajectories. This however is impossible, and hence no three of the equations can be independent.

Now (4) is of the form

$$L_r\, \theta'' + (a_r + Q_r)\, \theta'^2 = Y_r$$

where L_r is homogeneous and linear, Q_r homogeneous and quadratic, a_r and Y_r constant in the velocities; and since only two of the equations are independent, all the three rowed determinants of the matrix

$$\left\| \begin{matrix} a_1 + Q_1, & a_2 + Q_2, \ldots, & a_n + Q_n \\ L_1 \quad , & L_2 \quad , \ldots, & L_n \\ Y_1 \quad , & Y_2 \quad , \ldots, & Y_n \end{matrix} \right\|$$

must vanish identically.

Now any such determinant consists of two sets of terms, one linear and the other cubic in the velocities. Hence each set must vanish independently

of the other. Also the determinant of the coefficients of the L's does not vanish, since this determinant is the discriminant of the quadratic form $\mathfrak{T}$.

It follows at once that

$$\frac{a_1}{Y_1} = \frac{a_2}{Y_2} = \cdots = \frac{a_n}{Y_n} = A, \text{ say,}$$

and that $Q_r = b\,L_r + R\,Y_r$ $(r = 1, 2, \ldots, n)$ where b is linear, R quadratic in the magnitudes $\dot{x}$

We thus have the relations

$$\sum_{s=1}^{n} c_{rs}\,X^{(s)} = A\,Y_r \tag{5}$$

$$\sum_{p,q=1}^{n} k_{pqr}\,\dot{x}_p\,\dot{x}_q = \left(\sum_{p=1}^{n} b_p\,\dot{x}_p\right)\left(\sum_{s=1}^{n} c_{rs}\,\dot{x}_s\right) + Y_r \sum_{p,q=1}^{n} d_{pq}\,\dot{x}_p\,\dot{x}_q \tag{6}$$

$$(r = 1, 2, \ldots, n).$$

If we denote the system reciprocal to Y_r with respect to the form $\sum_{r,s=1}^{n} c_{rs}\,d\,x_r\,d\,x_s$ by $Y_1^{(r)}$, the equations (5) are equivalent to

$$X^{(r)} = Y_1^{(r)}\,A. \qquad (r = 1, 2, \ldots, n)$$

Now the lines of force of the system (1) are given by the equations

$$\frac{d\,x_1}{X^{(1)}} = \frac{d\,x_2}{X^{(2)}} = \cdots = \frac{d\,x_n}{X^{(n)}}.$$

We therefore have the result: *If two systems correspond the lines of force in the two systems correspond.*

In particular, if all the forces Y_r are zero, then all the quantities $X^{(s)}$, and therefore the forces X_r, are zero. Hence, *if the trajectories of one system are all geodesics, the trajectories of any corresponding system are also all geodesics.*

If the Y's are all zero, it is clear that no two of the equations (4) can be independent, and hence all the two rowed determinants of the matrix

$$\begin{Vmatrix} Q_1, & Q_2, \ldots, & Q_n \\ L_1, & L_2, \ldots, & L_n \end{Vmatrix}$$

must vanish, and therefore by the second lemma

$$Q_r = b\,L_r, \qquad (r = 1, 2, \ldots, n),$$

and therefore the equation (6) holds also in this case.

If we substitute from (5) and (6) in the equations (4) they give the two equations

$$\theta'' + \left(\sum_{p=1}^{n} b_p \dot{x}_p \right) \theta'^2 = 0 \tag{7}$$

$$\left(A + \sum_{p,q=1}^{n} d_{pq} \dot{x}_p \dot{x}_q \right) \theta'^2 = 1, \tag{8}$$

when the $Y's$ are not all zero, whilst in the case of geodesics they give the single equation (7).

From (6) we have by equating coefficients of the quantities $\dot{x}$,

$$c_{prq} + c_{qrp} - c_{pqr} = c_{rq} b_p + c_{rp} b_q + 2 Y_r d_{pq} \tag{9}$$

$$(p,\ q,\ r = 1,\ 2, \ldots,\ n).$$

Let a and c denote the discriminants of the two forms $2\,T$ and $2\,\mathfrak{T}$, and write $a\,k^{(rq)}$ for the cofactor of c_{pq} in c. Then from (9) we have immediately

$$\sum_{r=1}^{n} a\,k^{(rq)} (c_{rpq} + c_{rqp} - c_{pqr}) = c\,b_p + \frac{2\,c}{A} X^{(q)} d_{pq} \tag{10}$$

$$(p \neq q)$$

$$\sum_{r=1}^{n} a\,k^{(rp)} (c_{rpp} + c_{rpp} - c_{ppr}) = 2\,c\,b_p + \frac{2\,c}{A} X^{(p)} d_{pp}$$

$$(p,\ q = 1,\ 2, \ldots,\ n).$$

If we now sum over the values of q we have

$$a \sum_{r,q=1}^{n} k^{(rq)} c_{rqp} = (n+1)\,c\,b_p + \frac{2\,c}{A} \sum_{q=1}^{n} X^{(q)} d_{pq}$$

and it may be proved without difficulty that

$$\sum_{r,q=1}^{n} k^{(rq)} c_{rqp} = \frac{\partial}{\partial x_p} \left(\frac{c}{a} \right).$$

Hence we have the result

$$a\,(c/a)_p = (n+1)\,c\,b_p + \frac{2\,c}{A} \sum_{q=1}^{n} d_{pq} X^{(q)} \tag{11}$$

$$(p = 1,\ 2, \ldots,\ n).$$

Again, the equations (7) and (8) for θ must be compatible. Now $\frac{d}{dt}(\theta'^2) = 2\theta''$, and

$$\frac{d}{dt}\left\{A + \sum_{p,q=1}^{n} d_{pq}\,\dot{x}_p\,\dot{x}_q\right\} = \sum_{r=1}^{n} A_r\,\dot{x}_r + \sum_{p,q,r=1}^{n} d_{pqr}\,\dot{x}_p\,\dot{x}_q\,\dot{x}_r$$

$$+ 2\sum_{p,q=1}^{n} d_{pq}\,X^{(p)}\,\dot{x}_q,$$

and therefore if we differentiate (8) with respect to t, and substitute for θ'' from (7) we have the equation

$$\sum_{r=1}^{n} A_r\,\dot{x}_r + \sum_{p,q,r=1}^{n} d_{pqr}\,\dot{x}_p\,\dot{x}_q\,\dot{x}_r + 2\sum_{p,q=1}^{n} d_{pq}\,X^{(p)}\,\dot{x}_q$$

$$= 2\left[A + \sum_{p,q=1}^{n} d_{pq}\,\dot{x}_p\,\dot{x}_q\right]\left(\sum_{r=1}^{n} b_r\,\dot{x}_r\right)$$

which must be an identity.

Hence

$$2\,A\,b_r = A_r + 2\sum_{p=1}^{n} d_{pr}\,X^{(p)}, \qquad (r = 1,\ 2, \ldots,\ n) \tag{12}$$

$$d_{pqr} + d_{qrp} + d_{rpq} = 2\,(b_p\,d_{qr} + b_q\,d_{rp} + b_r\,d_{pq}, \qquad (p,\ q,\ r = 1,\ 2, \ldots,\ n) \tag{13}$$

If we substitute in (11) the value of $\sum d_{pr}\,X^{(p)}$ from (12) we have the result

$$\frac{\partial}{\partial x_p}\left(\log\frac{c}{a}\right) + \frac{\partial}{\partial x_p}(\log A) = (n+3)\,b_p,$$

and therefore b_p must be the derivative of a function b. If we write $b = \log\mu$. we have

$$\mu^{n+3} = \frac{c\,A}{a} \times \text{const.}$$

and without loss of generality we may take the constant to be unity. Thus

$$a\,\mu^{(n+3)} = c\,A, \quad \text{where} \quad b = \log\mu. \tag{14}$$

If the forces are all zero, equation (1) shows that b_p is the derivative of a function b, and in this case we have

$$a\,\mu^{n+1} = c.$$

In the general case, if we write $d_{pq} = \mu^2 \, e_{pq}$ (13) becomes

$$e_{pqr} + e_{qrp} + e_{rpq} = 0, \qquad (p, q, r = 1, 2, \ldots, n).$$

But these are the conditions that $\sum_{p,q=1}^{n} e_{pq} \dot{x}_p \dot{x}_q = \text{const.}$ be a quadratic integral of geodesics for the manifold for which $d s^2 = \sum_{r,s=1}^{n} a_{rs} \, d x_r \, d x_s$.

Again, from (7)

$$\frac{d}{dt} (\log \theta') + \frac{d}{dt} (\log \mu) = 0$$

along the selected trajectory, and therefore along this trajectory

$$\mu^2 \, \theta'^2 = \text{const.} = \frac{1}{C}, \quad \text{say.}$$

If this value of θ' be substituted in (8) we have

$$\frac{A}{\mu^2} + \sum_{p,q=1}^{n} e_{pq} \, \dot{x}_p \, \dot{x}_q = C$$

is a quadratic integral for trajectories.

Now write $A/\mu^2 = f$, and we have the following results:

If the system (1) *for which the forces are not all zero possesses a corresponding system it must admit the quadratic integral*

$$\sum_{p,q=1}^{n} e_{pq} \, \dot{x}_p \, \dot{x}_q + f = C \tag{15}$$

and the value of θ' is given by the equation

$$C \mu^2 \, \theta'^2 = 1, \quad \textit{where} \quad a \, \mu^{n+1} = c f.$$

For any trajectory

$$\mu^2 \, \theta'^2 \left\{ \sum_{p,q=1}^{n} e_{pq} \, \dot{x}_p \, \dot{x}_q + f \right\} = 1.$$

The magnitudes X are connected with f by the relations

$$f_r + 2 \sum_{p=1}^{n} e_{pr} \, X^{(p)} = 0, \qquad (r = 1, 2, \ldots, n) \tag{16}$$

and in addition the coefficients c_{rs} *and the forces* Y_r *must satisfy the set of equations*

$$\sum_{s=1}^{n} c_{rs} X^{(s)} = \mu^2 f Y_r \qquad (r = 1, 2, \ldots, n)$$

which show that the lines of force of the two systems correspond, and also the set

$$c_{pqr} + c_{qrp} - c_{pqr} = c_{rp} b_q + c_{rq} b_p + 2 Y_r \mu^2 e_{pq} \tag{9}$$

$$(p, q, r = 1, 2, \ldots, n)$$

where $b = \log \mu$. *These are the necessary and sufficient conditions for correspondence of the two systems* (1) *and* (2).

In the case where the forces X *are all zero the* Y'*s must also be zero; we have*

$$a \mu^{n+1} = c, \qquad C \mu^2 \theta'^2 = 1$$

and

$$c_{prp} + c_{qrp} - c_{pqr} = c_{rq} b_p + c_{rp} b_q$$

$$(p, q, r = 1, 2, \ldots, n).$$

These again are the necessary and sufficient conditions for correspondence.

If the forces are not zero, but the magnitudes e_{pq} are all zero, it follows readily that f is unity, the conditions are satisfied for correspondence of geodesics, and $\sum_{s=1}^{n} c_{rs} X^{(s)} = \mu^2 Y_r$, $(r = 1, 2, \ldots, n)$.

Conversely, *if the two manifolds for which the fundamental forms are*

$$\sum_{r,s=1}^{n} a_{rs}\, d x_r\, d x_s \quad \textit{and} \quad \sum_{r,s=1}^{n} c_{rs}\, d x_r\, d x_s$$

are such that their geodesics correspond, the systems (1) *and* (2) *will correspond, provided*

$$\sum_{s=1}^{n} c_{rs} X^{(s)} = \mu^2 Y_r f, \qquad (r = 1, 2, \ldots, n)$$

where $\mu^{n+1} = c/a$.

§ 4.

We now choose the canonical orthogonal ennuple for the associated form $\sum_{r,s=1}^{n} c_{rs}\, d\, x_r\, d\, x_s$. Then

$$a_{rs} = \sum_{h=1}^{n} \lambda_{h|r}\, \lambda_{h|s}$$

$$c_{rs} = \sum_{h=1}^{n} \rho_h\, \lambda_{h|r}\, \lambda_{h|s}$$

and in terms of this ennuple the quantities e_{rs} may be written

$$e_{rs} = \sum_{p,q=1}^{n} \sigma_{pq}\, \lambda_{p|r}\, \lambda_{q|s}\,.$$

The quantities σ_{pq} are invariants, and $\sigma_{pq} = \sigma_{qp}$. Also if F is any function of the coordinates

$$\sum_{r=1}^{n} \lambda_h^{(r)} \frac{\partial F}{\partial x_r} = \frac{\partial F}{\partial s_h}$$

denotes the rate of change of F as we proceed along an arc of the congruence h.

Also, the rule for covariant differentiation of a product is the same as that for ordinary differentiation, and hence

$$c_{pqr} = \sum_{h=1}^{n} (\rho_{h|r}\, \lambda_{h|p}\, \lambda_{h|q} + \rho_h\, \lambda_{h|pr}\, \lambda_{h|q} + \rho_h\, \lambda_{h|p}\, \lambda_{h|qr})$$

$$= \sum_{h=1}^{n} \rho_{h|r}\, \lambda_{h|p}\, \lambda_{h|q} + \sum_{h,i,j=1}^{n} (\gamma_{hij}\, \rho_h\, \lambda_{i|p}\, \lambda_{j|r}\, \lambda_{h|q} + \gamma_{hij}\, \rho_h\, \lambda_{h|p}\, \lambda_{i|q}\, \lambda_{j|r}).$$

If the values in terms of the λ's of the various functions involved be substituted in the typical equation (9) and if this typical equation be multiplied by $\lambda_h^{(p)}\, \lambda_i^{(q)}\, \lambda_j^{(r)}$ and a summation made over p, q, r we obtain the fol-

lowing set of equations, which are equivalent to the set (9)

$$\text{(i)} \ldots \ (\rho_i - \rho_h)\gamma_{ihj} + (\rho_j - \rho_h)\gamma_{jhi} - (\rho_i - \rho_j)\gamma_{ijh} = 2\mu^2 \frac{\partial Y}{\partial s_h}\sigma_{ij}, \qquad (17)$$

$$(h \neq i \neq j)$$

$$\text{(ii)} \ldots \quad \frac{\partial \rho_h}{\partial s_i} = \frac{\partial b}{\partial s_i}\rho_h + 2\mu^2 \frac{\partial Y}{\partial s_h}\sigma_{hi}, \quad (h \neq i)$$

$$\text{(iii)} \ldots \quad 2(\rho_h - \rho_i)\gamma_{hii} - \frac{\partial \rho_i}{\partial s_h} = 2\mu^2 \frac{\partial Y}{\partial s_h}\sigma_{hi}, \quad (h \neq i)$$

$$\text{(iv)} \ldots \quad \frac{\partial \rho_h}{\partial s_h} = 2\rho_h \frac{\partial b}{\partial s_h} + 2\mu^2 \frac{\partial Y}{\partial s_h}\sigma_{hh},$$

$$(h,\ i,\ j = 1,\ 2, \ldots,\ n).$$

In these equations $\frac{\partial Y}{\partial s_h}$ is written for $\sum_{r=1}^{n} \lambda_h^{(r)} Y_r$. When the system of equations

$$\sum c_{rs} X^{(s)} = \mu^2 f Y_r$$

are thus written in invariantive form they become

$$\rho_h \frac{\partial X}{\partial s_h} = \mu^2 f \frac{\partial Y}{\partial s_h}, \quad (h = 1,\ 2, \ldots,\ n) \qquad (18)$$

where

$$\frac{\partial X}{\partial s_h} \equiv \sum_{r=1}^{n} \lambda_h^{(r)} X_r.$$

and the equations (16) are equivalent to

$$\frac{\partial f}{\partial s_h} + 2\sum_k \sigma_{hk} \frac{\partial X}{\partial s_k} = 0. \qquad (19)$$

The only remaining equations to be satisfied are those that express that $\sum_{r,s=1}^{n} e_{rs}\dot{x}_r\dot{x}_s = \text{const.}$ is a quadratic integral of geodesics. These are

$$\frac{\partial \sigma_{hi}}{\partial s_j} + \frac{\partial \sigma_{ij}}{\partial s_h} + \frac{\partial \sigma_{jh}}{\partial s_i} + \sum_{p=1}^{n}\left(\sigma_{pi}\,\overline{\gamma_{phj} + \gamma_{pjh}} + \sigma_{pj}\,\overline{\gamma_{phi} + \gamma_{pih}} + \sigma_{ph}\,\overline{\gamma_{pij} + \gamma_{pji}}\right), \qquad (20)$$

$$(h,\ i,\ j = 1,\ 2, \ldots,\ n).$$

We consider the particular case for which $\sigma_{ij} = 0$ if $i \neq j$. In this case the equations [17. (i)] give

$$(\rho_i - \rho_j)\,\gamma_{ijh} = 0, \qquad (h \neq i \neq j)$$

and therefore $\gamma_{ijh} = 0$ if $\rho_i \neq \rho_j$.

Suppose that

$$\begin{aligned} \rho_1 = \quad \rho_2 = \cdots = \rho_\alpha &\equiv \rho^{(1)} \\ \rho_{\alpha+1} = \rho_{\alpha+2} = \cdots = \rho_\beta &\equiv \rho^{(2)} \\ \rho_{\beta+1} = \rho_{\beta+2} = \cdots = \rho_\gamma &= \rho^{(3)} \end{aligned}$$

and so on. Then the numbers from 1 to n are divided into sets and, $(h \neq i,\ h \neq j)$, we have $\gamma_{ijh} = 0$ if i and j do not belong to the same set.

Now, F being any function, $\frac{\partial F}{\partial s_i}$ is a linear function of the derivatives of F, and it is not difficult to prove that

$$\frac{\partial}{\partial s_k}\left(\frac{\partial F}{\partial s_h}\right) - \frac{\partial}{\partial s_h}\left(\frac{\partial F}{\partial s_k}\right) = \sum_{\iota=1}^{n} (\gamma_{ikh} - \gamma_{ihk})\,\frac{\partial F}{\partial s_i}\,.$$

If we suppose h and k to belong to the first set we have

$$\frac{\partial}{\partial s_k}\left(\frac{\partial F}{\partial s_h}\right) - \frac{\partial}{\partial s_h}\left(\frac{\partial F}{\partial s_k}\right) = \sum_{\iota=1}^{\alpha} (\gamma_{ikh} - \gamma_{ihk})\,\frac{\partial F}{\partial s_i}\,,$$

since all the remaining coefficients on the right vanish, and therefore the set of equations

$$\frac{\partial F}{\partial s_1} = 0 \cdot \frac{\partial F}{\partial s_2} = 0 \cdots \frac{\partial F}{\partial s_\alpha} = 0$$

is complete.

Similarly each other set associated with a given ρ is complete. Also it is clear that the whole set of n equations $\frac{\partial F}{\partial s_i} = 0\,.\ (\iota = 1, 2, \ldots, n)$ have no common solutions.

The system $\frac{\partial F}{\partial s_i} = 0$ $(i = \alpha + 1, \ldots, n)$ possesses α functionally independent solutions. Let these be chosen as the variables $x_1, \ldots, x_\alpha$, and choose the remaining $n - \alpha$ variables as the fnnctionally independent solutions of $\frac{\partial F}{\partial s_i} = 0$, $(i = 1, 2, \ldots, \alpha)$. We thus have a set of variables $x_1, x_2, \ldots, x_\alpha$ as-

sociated with $\rho^{(1)}$. It is clear that we may at the same time choose a set $x_{\alpha+1},\ldots, x_\beta$ associated with $\rho^{(2)}$, and so on.

With this choice of variables we have $\lambda_p^{(r)} = 0$, $\lambda_{p|r} = 0$, if p and r do not belong to the same set. It at once follows that a_{rs}, c_{rs}, e_{rs} are all zero if r and s belong to different sets. We need generally to speak of the variables of, say, the k^{th} set as a whole. We shall call these variables y_k; thus, if we say that F is a function of y_k only, we mean that it involves only the variables of the k^{th} set. Also we write σ_p for σ_{pp}.

From the equations (20) it readily follows that

$$\sigma_1 = \sigma_2 = \sigma_3 = \cdots = \sigma_\alpha,$$

$$\sigma_{\alpha+1} = \sigma_{\alpha+2} = \cdots = \sigma_\beta,$$

and so on.

There is thus a quantity σ associated with each set. If the σ associated with the k^{th} set be called $\sigma^{(k)}$ it follows also from (20) that $\sigma^{(k)}$ is independent of y_k. Let t belong to the k^{th} set, them we have

$$\sum_{\lambda=1}^{n} \{r s, \lambda\} e_{\lambda t} = \sum_{\lambda} \{r s, \lambda\} \sigma^{(k)} a_{\lambda t} = \sigma^{(k)} [r s, t].$$

Since $\sum\limits_{p,q=1}^{n} e_{pq} \dot{x}_p \dot{x}_q = \text{const.}$ is a quadratic integral for geodesics we have $e_{rst} + e_{str} + e_{trs} = 0$. $(r, s, t = 1, 2,\ldots, n)$ and it readily follows that, if r, s, t belong to the same set, these equations are satisfied identically in virtue of the conditions already imposed.

The equations are also satisfied if r, s, t belong all to different sets. Now suppose that r, s belong to the k^{th} set, t to the i^{th} set where $i \neq k$. Then

$$\frac{\partial e_{rs}}{\partial x_t} - \sigma^{(k)} [r t, s] - \sigma^{(k)} [s t, r] - \sigma^{(i)} [s r, t] - \sigma^{(k)} [t r, s] -$$

$$- \sigma^{(k)} [s t, r] - \sigma^{(i)} [r s, t] = 0.$$

or

$$\frac{\partial}{\partial x_t} \log (\sigma^{(k)} - \sigma^{(i)}) = \frac{\partial}{\partial x_t} (\log a_{rs}).$$

Hence it follows that $a_{rs} = a^{(k)} . \alpha_{rs}$, where α_{rs} is a function of y_k only,

and

$$\frac{\sigma^{(k)} - \sigma^{(i)}}{a^{(k)}} \quad \text{does not involve } y_i\,.$$

The quantities $\sigma^{(k)}$ and $a^{(k)}$ must therefore satisfy the functional equations

$$\frac{\sigma^{(k)} - \sigma^{(\iota)}}{a^{(k)}} = \text{a function independent of } y_i\,,$$

$$\sigma^{(\iota)} = \text{a function independent of } y_\iota\,.$$

These functional equations are practically these solved by Di Pirro (*). Their complete solution in given by Painlévé (**).

This complete solution is the following.

Let

$$\Delta = \begin{vmatrix} \varphi_1^{(1)}(y_1), & \varphi_2^{(1)}(y_2), \ldots, & \varphi_q^{(1)}(y_q) \\ \varphi_1^{(2)}(y_1), & \varphi_2^{(2)}(y_2), \ldots, & \varphi_q^{(2)}(y_q) \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ \varphi_1^{(q)}(y_1), & \varphi_2^{(q)}(y_2), \ldots, & \varphi_q^{(q)}(y_q) \end{vmatrix},$$

where φ_k is a function of y_k only, then if $\Delta_r^{(s)}$ denote the cofactor of the element in the s^{th} now and r^{th} column of Δ,

$$a^{(k)} = \frac{\Delta}{\Delta_k^{(1)}}, \qquad \sigma^{(k)} = \frac{\Delta_k^{(2)}}{\Delta_k^{(1)}}\,.$$

We still have to satisfy equations 17 (ii), (iii), (iv), 18, 19. From 17 (ii) it follows at once that

$$\rho^{(k)} = \xi_k\,\mu,$$

where ξ_k is a function of y_k only. If the k^{th} set contains more than one variable x, these equations also show that ξ_k is an absolute constant. Otherwise if the set contains one variable, ξ_k is a function, as yet arbitrary, of that variable.

We now go back to the equivalent set of equations (9). We have $c_{p,q} = 0$

(*) *Annali di Mat.*, Ser. II, Tom. XXIV (1896).

(**) *Comptes Rendus* (1897) 1st Feb., pp. 221 sqq. See also Stäckel, *C. R.*, 9th Mar. 1893, 7 Oct. 1895.

if p and q belong to different sets, and if p and q belong to the k^{th} set $c_{pq} = \rho^{(k)} a_{pq}$; also $a_{pq} = a^{(k)} \alpha_{pq}$ where α_{pq} is a function of y_k only. These equations (9) are

$$\frac{\partial}{\partial x_q}(c_{rp}) + \frac{\partial}{\partial x_p}(c_{rq}) - \frac{\partial}{\partial x_r}(c_{pq}) -$$

$$- 2 \sum_{\lambda=1}^{n} \{p\, q,\ \lambda\}\, c_{r\lambda} = c_{rp}\, b_q + c_{rq}\, b_q + 2\, Y_r\, \mu^2\, e_{pq}.$$

(Suppose r to belong to the k^{th} set, then

$$2 \sum_{\lambda=1}^{n} \{p\, q,\ \lambda\}\, c_{r\lambda} = 2\, \rho^{(k)}\, [p\, q,\ r]\Big).$$

If p, q, r all belong to the k^{th} set the equation gives

$$a_{rp} \frac{\partial \rho^{(k)}}{\partial x_q} + a_{rq} \frac{\partial \rho^{(k)}}{\partial x_p} - a_{pq} \frac{\partial \rho^{(k)}}{\partial x_r} =$$

$$= a_{rp}\, \rho^{(k)} \,.\, b_q + a_{rq}\, \rho^{(k)}\, b_p + 2\, Y_r\, \mu^2\, \sigma^{(k)}\, a_{pq}.$$

Assume this set to contain more than one variable x, then ξ_k is a constant and this equation gives

$$2\, Y_r\, \mu^2\, \sigma^{(k)}\, a_{pq} + a_{pq}\, \rho^{(k)}\, b_r = 0.$$

or

$$2\, Y_r\, \mu\, \sigma^{(k)} + \xi_k\, b_r = 0.$$

If on the other hand the set contains only one letter, say p, the above equation gives

$$\frac{\partial \rho^{(k)}}{\partial x_p} = 2\, b_p\, \rho^{(k)} + 2\, Y_r\, \mu^2\, \sigma^{(k)}$$

or

$$2\, Y_p\, \mu\, \sigma^{(k)} + \xi_k\, b_p = \frac{\partial \xi_k}{\partial x_p}.$$

Hence in either case

$$2\, Y_r\, \sigma^{(k)} = \frac{\partial}{\partial x_r}\left(\frac{\xi_k}{\mu}\right) \qquad (r \text{ belongs to } k^{th} \text{ set}) \qquad (21)$$

and the equations (9) for which p, q, r belong to the same set are equivalent to the equations (21). If p and q belong to different sets it is easy to

see that the corresponding equations (9) are satisfied identically in virtue of the conditions already imposed.

Suppose finally that p, q both belong to the i^{th} set, r to the k^{th} set, where $k \neq i$.

Then (9) gives

$$-\frac{\partial}{\partial x_r}(\rho^{(i)} a_{pq}) + \rho^{(k)} \frac{\partial a_{pq}}{\partial x_r} = 2 Y_r \mu^2 \sigma^{(i)} a_{pq},$$

or, since α_{pq} does not involve x_r,

$$-\frac{\partial}{\partial x_r}(\rho^{(i)} a^{(i)}) + \rho^{(k)} \frac{\partial a^{(i)}}{\partial x_r} = 2 Y_r \mu^2 \sigma^{(i)} a^{(i)}$$

$$= \mu^2 \frac{\sigma^{(i)}}{\sigma^{(k)}} a^{(i)} \frac{\partial}{\partial x_r}\left(\frac{\xi_k}{\mu}\right)$$

in virtue of (21).

The left side of this equation is

$$(\rho^{(k)} - \rho^{(i)}) \frac{\partial a^{(i)}}{\partial x_r} - a^{(i)} \frac{\partial \rho^{(i)}}{\partial x_r},$$

or, since $\rho^{(i)} = \xi_i \mu$, and ξ_i does not involve x_r, it is

$$\mu^2 \left\{ \frac{\xi_k - \xi_i}{\mu} \frac{\partial a^{(i)}}{\partial x_r} + a^{(i)} \frac{\partial}{\partial x_r}\left(\frac{\xi_i}{\mu}\right) \right\}$$

$$= -\mu^2 a^{(i)2} \left\{ \frac{\partial}{\partial x_r}\left(\frac{\xi_k - \xi_i}{\mu}\right) \right\} + \mu^2 a^{(i)} \frac{\partial}{\partial x_r}\left(\frac{\xi_k}{\mu}\right).$$

Hence the above equation becomes

$$-\frac{\partial}{\partial x_r}\left(\frac{\xi_k - \xi_i}{\mu\, a^{(i)}}\right) = \frac{\sigma^{(i)} - \sigma^{(k)}}{a^{(i)} \sigma^{(k)}} \frac{\partial}{\partial x_r}\left(\frac{\xi_k}{\mu}\right).$$

Also $\dfrac{\sigma^{(i)} - \sigma^{(k)}}{a^{(i)}}$ and $\sigma^{(k)}$ are each of them independent of y_k, and thus independent of x_r. It therefore follows by integration that

$$\frac{\xi_k - \xi_i}{\mu\, a^{(i)}} - \frac{\sigma^{(k)} - \sigma^{(i)}}{\sigma^{(k)} a^{(i)}} \frac{\xi_k}{\mu}$$

is independent of x_r, and therefore of the set of variables y_k.

If we replace $a^{(i)}$, $\sigma^{(k)}$, $\sigma^{(i)}$ by their values in terms of the cofactors of Δ, this becomes

$$\frac{\xi_k\, \Delta_i^{(2)}\, \Delta_k^{(1)} - \xi_i\, \Delta_k^{(2)}\, \Delta_i^{(1)}}{\mu\, \Delta\, \Delta_k^{(2)}},$$

and since $\Delta_k^{(2)}$ is independent of y_k it follows that if

$$\xi_k\, \Delta_i^{(2)}\, \Delta_k^{(1)} - \xi_i\, \Delta_k^{(2)}\, \Delta_i^{(1)} = \mu\, \Delta\, F_{ki} \tag{22}$$

then F_{ki} is independent of y_k. Also F_{ki} obviously is equal to $-F_{ik}$, and hence F_{ki} in also independent of y_i. It follows that the equations still to be satisfied are equivalent to (22), where k and i take all values from 1 to q, and F_{ki} is a function independent of both y_k and y_i.

The solution of our problem is now reduced to the solution of the functional equations (22). When these are solved, (21) gives the values of the forces Y, and then from (18) we have the forces X. The unknown function f may be determined at once from (18), (19), and (21), for from (19) $\frac{\partial f}{\partial s_h} = -2\,\sigma_h \frac{\partial X}{\partial s_h}$, and from (18) $\rho_h \frac{\partial X}{\partial s_h} = \mu^2 f \frac{\partial Y}{\partial s_h}$. Now if h belongs to the k^{th} set we have from (21), $2\,\sigma_h \frac{\partial Y}{\partial s_h} = \frac{\partial}{\partial s_h}\left(\frac{\xi_k}{\mu}\right)$, and it follows without difficulty that

$$\frac{\partial}{\partial s_h} \cdot \log\left(\frac{f\, \xi_k}{\mu}\right) = 0,$$

Hence $\log \frac{f\, \xi_k}{\mu}$ is independent of y_k, for all values of k, and therefore

$$\frac{f\, \xi_1\, \xi_2 \ldots \xi_q}{\mu}$$

is an absolute constant, say C, and, finally,

$$f = C\,\mu \,/\, \xi_1\, \xi_2 \ldots \xi_q\,.$$

We now proceed to solve the functional equations (22). These equations give at once

$$\Delta_h^{(2)}\, F_{ki} + \Delta_k^{(2)}\, F_{ih} + \Delta_i^{(2)}\, F_{hk} = 0$$

for all values of h, k, i. Also

$$\sum_r \Delta_r^{(2)}\, \varphi_r^{(3)} = 0, \qquad (s = 1, 3, 4, \ldots, q).$$

It follows that quantities l can be found such that

$$F_{ki} = \sum_{r=1}^{q} l_r \varphi_h^{(r)}, \tag{α}$$

$$F_{ih} = \sum_{r=1}^{q} l_r \varphi_k^{(r)}, \tag{β}$$

$$F_{hk} = \sum_{r=1}^{q} l_r \varphi_i^{(r)}, \tag{γ}$$

$$0 = \sum_{r=1}^{q} l_r \varphi_s^{(r)}, \tag{δ}$$

where l_2 is zero, and s takes all values from 1 to q with the exception of h, k, i.

Let x be one of the variables of the set h, and differentiate (β), (γ), and the equations (δ) with respect to x. We thus obtain the $n-1$ equations

$$0 = \sum_{r=1}^{q} \frac{\partial l_r}{\partial x} \varphi_s^{(r)}, \qquad (s \neq h),$$

in the $(n-1)$ variables $\frac{\partial l_r}{\partial x}\cdot$ The determinant of these equations is $\Delta_h^{(2)}$. We make the assumption that none of the determinants $\Delta^{(2)}$ vanish, and it follows that the $l's$ are independent of x, and finally that the $l's$ are independent of the sets of variables h, k, i.

We conclude without difficulty that F_{hk} for example is a function linear in each of the sets of variables

$$\varphi_s^{(1)}, \quad \varphi_s^{(3)}, \quad \varphi_s^{(4)}, \ldots, \quad \varphi_s^{(q)}$$

where s takes all values from 1 to q excluding h and k.

Now let $F_{hk} = \sum_{ij} l_{ij} \varphi_\alpha^{(i)} \varphi_\beta^{(j)}$ then l_{ij} is independent of the variables y_h, y_k, y_σ, y_β, and

$$F_{h\alpha} = \sum_{ij} l_{ij} \varphi_h^{(i)} \varphi_\beta^{(j)}$$

and hence

$$F_{\alpha\beta} = \sum_{ij} l_{ij} \varphi_h^{(i)} \varphi_k^{(j)}.$$

Similarly

$$F_{k\beta} = \sum_{ij} l_{ij} \varphi_\alpha^{(i)} \varphi_h^{(j)}$$

and

$$F_{\beta\alpha} = \sum_{ij} l_{ij}\, \varphi^{(i)}_k\, \varphi^{(j)}_h = \sum_{ij} l_{ji}\, \varphi^{(i)}_h\, \varphi^{(j)}_k .$$

Now $F_{\alpha\beta} + F_{\beta\alpha} = 0$, and therefore $l_{ij} + l_{ji} = 0$. It thus follows that

$$F_{hk} = \sum_{i,j} l_{ij} \begin{vmatrix} \varphi^{(i)}_\alpha & \varphi^{(j)}_\alpha \\ \varphi^{(i)}_\beta & \varphi^{(j)}_\beta \end{vmatrix},$$

where the summation extends only once to each pair of letters i, j.

Again, write $l_{ij} = \sum\limits_r l_{ijr}\, \varphi^{(r)}_\gamma$. We see that l_{ijr} is independent of y_γ, and deduce readily that

$$F_{hk} = \sum_{i,j,r} l_{ijr} \begin{vmatrix} \varphi^{(i)}_\alpha & \varphi^{(j)}_\alpha & \varphi^{(r)}_\alpha \\ \varphi^{(i)}_\beta & \varphi^{(j)}_\beta & \varphi^{(r)} \\ \varphi^{(i)}_\gamma & \varphi^{(j)}_\gamma & \varphi^{(r)}_\gamma \end{vmatrix},$$

where the summation extends only once to each set of letters i, j, r. Continuing thus, we deduce finally that

$$F_{hk} = \sum_{r=1}^{q} a_r\, \Delta^{(2r)}_{hk},$$

where the $a's$ are absolute constants, and $\Delta^{(2r)}_{hk}$ is the second minor of Δ, with appropriate sign, obtained by omitting the second and r^{th} rows, and the h^{th} and k^{th} columna.

Now, from the theory of determinants we have

$$\Delta\, \Delta^{(2r)}_{hk} = \Delta^{(2)}_h\, \Delta^{(r)}_k - \Delta^{(2)}_k\, \Delta^{(r)}_h ,$$

and hence (22) becomes

$$\xi_k\, \Delta^{(2)}_i\, \Delta^{(1)}_k - \xi_i\, \Delta^{(2)}_k\, \Delta^{(1)}_i = \mu \sum_{r=1}^{q} a_r \left\{ \Delta^{(2)}_k\, \Delta^{(r)}_i - \Delta^{(2)}_i\, \Delta^{(r)}_k \right\}$$

or

$$\frac{\xi_k\, \Delta^{(1)}_k + \mu \sum a_r\, \Delta^{(r)}_k}{\Delta^{(2)}_k} = \frac{\xi_i\, \Delta^{(1)}_i + \mu \sum a_r\, \Delta^{(r)}_i}{\Delta^{(3)}_i} = -P, \text{ say.}$$

Hence

$$\xi_k\, \Delta^{(1)}_k + \mu \sum_{r=1}^{q} a_r\, \Delta^{(r)}_k + P\, \Delta^{(2)}_k = 0, \qquad (k = 1, 2, \ldots, n)$$

and a_2 is zero.

This equation is, for example when $k = 1$

$$\begin{vmatrix} \xi_1 + \mu\, a_1, & \varphi_2^{(1)}, & \varphi_3^{(1)}, & \ldots, & \varphi_q^{(1)} \\ P \quad, & \varphi_2^{(2)}, & \varphi_3^{(2)}, & \ldots, & \varphi_q^{(2)} \\ \mu\, a_3 \quad, & \varphi_2^{(3)}, & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \mu\, a_q \quad, & \varphi_2^{(q)}, & \ldots & \ldots & \varphi_q^{(q)} \end{vmatrix} = 0.$$

We write $a_3\, \varphi_r^{(3)}$ instead of $\varphi_r^{(3)}$ in Δ, and then write $\varphi_r^{(s)} + a_s\, \varphi_r^{(3)}$ instead of $\varphi_r^{(s)}$ in Δ. These changes do not affect Δ, $\Delta^{(1)}$, $\Delta^{(2)}$, and our equation now takes the form

$$\xi_k\, \Delta_k^{(1)} + \mu\, \Delta_k^{(3)} + P\, \Delta_k^{(2)} = 0, \qquad (k = 1,\ 2, \ldots,\ q)$$

where the minors are those of the determinant Δ of the new functions φ.

We multiply this equation by $\varphi_k^{(s)}$, and sum over the values of k, and thus obtain the q equivalent equations

$$\sum_{k=1}^{q} \xi_k\, \varphi_k^{(2)}\, \Delta_k^{(1)} + P\, \Delta = 0,$$

$$\sum_{k=1}^{q} \xi_k\, \varphi_k^{(3)}\, \Delta_k^{(1)} + \mu\, \Delta = 0,$$

$$\sum_{k=1}^{q} \xi_k\, \varphi_k^{(s)}\, \Delta_k^{(1)} \qquad = 0, \qquad (s \neq 2,\ 3).$$

also if $s \neq 1$, $\sum_{k=1}^{q} \varphi_k^{(s)}\, \Delta_k^{(1)} = 0$.

And it follows that

$$\xi_k\, \varphi_k^{(s)} = \sum_{t=2}^{q} m_{st}\, \varphi_k^{(t)}, \qquad (s \neq 2,\ 3)$$

for all values of k.

Differentiate both sides of this equation with respect to x, where x is a variable of the set h, and h is not equal to k. There is thus obtained the equation

$$0 = \sum_{t=r}^{q} \frac{\partial\, m_{st}}{\partial\, x}\, \varphi_k^{(t)}.$$

We have $(q - 1)$ equations obtained by giving k all values from 1 to q, except h, and keeping s fixed. The determinant of these equations is $\Delta_h^{(1)}$,

which does not vanish, and hence $\frac{\partial m_{st}}{\partial x} = 0$. It thus follows easily that the $m's$ are all absolute constants, and we have the $q-2$ equations

$$\zeta_k\, \varphi_k^{(s)} = \sum_{t=2}^{q} m_{st}\, \varphi_k^{(t)}, \qquad (s \neq 2,\ 3).$$

among the $q+1$ functions of y_k,

$$\zeta_k, \quad \varphi_k^{(1)}, \quad \varphi_k^{(2)}, \ldots, \quad \varphi_k^{(q)}.$$

We thus have the complete solution of our problem in this special case, and μ is given from the equation

$$\mu\,\Delta = -\sum_{k=1}^{q} \zeta_k\, \varphi_k^{(3)}\, \Delta_k^{(1)}.$$

We thus have corresponding systems in n variables given as follows: Let the variables be divided into q sets and let $\tau_h = \frac{1}{2} \sum \alpha_{rs}\, d\,x_r\, d\,x_s$ be a quadratic differential form in the h^{th} set of variables only. Also let this set of variables as a whole be called y_h. If any set h contains more than one variable choose a constant ζ_h associated with this set. If it contains only one variable choose ζ_h an arbitrary function of that variable. Take a number of arbitrary constants m_{st}, where s takes all values from 1 to q except 2, 3, and t takes all values from 2 to q. Let $\varphi_k^{(r)}$ denote a function of y_k only, and choose a set of functions $\varphi_k^{(r)}$ which satisfy

$$\zeta_k\, \varphi_k^{(s)} = \sum_{t=2}^{q} m_{st}\, \varphi_k^{(t)}, \qquad (s = 1,\ 4,\ 5, \ldots,\ q)$$

but are otherwise arbitrary.

Let Δ be the determinant $|\varphi_k^{(r)}|$ of these functions, and let the cofactors of its elements be denoted by $\Delta_k^{(r)}$. Also let a function μ be given by the equation

$$-\mu\,\Delta = \sum_{k=1}^{q} \zeta_k\, \varphi_k^{(3)}\, \Delta_k^{(1)}$$

and let

$$a^{(k)} = \frac{\Delta}{\Delta_k^{(1)}}, \qquad c^{(k)} = \zeta_k\, \mu\, a^{(k)}.$$

Then if

$$T\,d\,t^2 = \sum_{k=1}^{q} a^{(k)}\,\tau_k\,, \qquad \mathfrak{T}\,d\,t_1^2 = \sum_{k=1}^{q} c^{(k)}\,\tau_k\,,$$

the two systems (1) and (2) correspond, and the forces X_r, Y_r of the two systems are given by

$$2\,\frac{\Delta_k^{(2)}}{\Delta_k^{(1)}}\,Y_r = \frac{\partial}{\partial\,x_r}\left(\frac{\xi_k}{\mu}\right), \qquad \xi_k\,X_r = \mu\,f\,Y_r\,,$$

where r belongs to the k^{th} set of variables, and

$$\frac{\mu}{f} = C\,\xi_1\,\xi_2\ldots\xi_q$$

where C is a constant.

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SOME SURFACES HAVING A FAMILY OF HELICES AS ONE SET OF LINES OF CURVATURE.*

BY MISS EVA M. SMITH.

In a recent paper, Forsyth † gives a general method for the determination of surfaces with assigned lines of curvature, and he solves completely the case where both sets are circles. We apply the method to the case where one of the given sets consists of helices, and it appears that surfaces do exist having as one set of lines of curvature general helices ($\rho/\tau =$ constant along each curve), but there are always limitations on the forms of ρ and τ. In particular, ρ and τ cannot both be constant along every curve. The complete solution seems to be too wide for analytic discussion, but there are two particular cases for which definite results can be obtained. This note contains a discussion of these cases.

1) Assuming that ρ and τ are constant along each curve of the set (regular helices), we obtain the result that: *There are no surfaces with regular helices as one set of lines of curvature.*

2) If ρ/τ is constant along each curve of one set of lines of curvature, and the other set consists of geodesics, we can obtain a complete solution ; the equations of the resulting surfaces in parametric form are given at the end of this paper. The notation and equations used are those given in Darboux, Théorie des surfaces, volume 2, but derivatives with respect to u and v are here denoted by suffixes 1 and 2 respectively.‡

§ 1.

Consider the case where the helices are all regular. The curves $v =$ constant are helices, and therefore ρ and τ for these curves are functions of v only, and we denote τ/ρ by k.

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* For the suggestion of this subject I am indebted to Prof. A. R. Forsyth.

† *Messenger of Mathematics*, vol. 38 (1908), pp. 33–44.

‡ Note that p_1 is not $\partial p/\partial u$. To express derivatives of the rotations we use parentheses, *e. g.* $(p)_1 = \partial p/\partial u$.

We have the following set of equations* :

$$p = 0, \quad q_1 = 0,$$

$$(1) \quad \frac{1}{\tau} = \frac{1}{A}\phi_1,$$

$$(2) \quad r = \frac{\sin\phi}{\rho}A, \qquad (2') \quad r = -\frac{1}{C}A_2,$$

$$(3) \quad q = -\frac{\cos\phi}{\rho}A, \qquad (4) \quad p_1 = \frac{1}{r}(q)_2,$$

$$(5) \quad r_1 = -\frac{1}{q}(p_1)_1, \qquad (5') \quad r_1 = \frac{1}{A}C_1,$$

where ϕ is the angle between the osculating plane of the curve $v =$ constant and the normal plane to the surface. From (2), (3), (4), (5) we obtain

$$(7) \quad r_1 = 2\frac{\phi_{12}}{k\phi_1}\sec\phi + \frac{k_2}{k^2}\operatorname{cosec}^2\phi\sec\phi - \frac{1}{k\phi_1}\left(\frac{\phi_{12}\operatorname{cosec}\phi}{\phi_1}\right)_1$$

Also from (1), (2′), (5′),

$$(8) \qquad r_1 = \frac{1}{k}\left[\frac{\tau_2}{\tau}\cot\phi\operatorname{cosec}\phi - \frac{1}{\phi_1}\left(\frac{\phi_{12}\operatorname{cosec}\phi}{\phi_1}\right)_1\right].$$

From (7) and (8)

$$\frac{\phi_{12}}{\phi_1} = \tfrac{1}{2}\left(\frac{\tau_2}{\tau}\cot^2\phi - \frac{k_2}{k}\operatorname{cosec}^2\phi\right).$$

Differentiating with respect to u and substituting in (8), we get

$$(9) \qquad r_1 = \tfrac{3}{2}\cot\phi\operatorname{cosec}\phi\left[\operatorname{cosec}^2\phi\left(\frac{\tau_2}{\tau} - \frac{k_2}{k}\right) + \frac{\tau_2}{\tau}\right].$$

Combining (9) with (6) we obtain, after some reduction,

$$6\operatorname{cosec}^4\phi\left(\frac{\tau_2}{\tau} - \frac{k_2}{k}\right) - \tfrac{3}{2}\operatorname{cosec}^2\phi\left(\frac{\tau_2}{\tau} - \frac{3k_2}{k}\right)$$
$$+\frac{k^2}{2}\left[\operatorname{cosec}^2\phi\left(\frac{\tau_2}{\tau} - \frac{k_2}{k}\right) - \frac{\tau_2}{\tau}\right] = 0.$$

* Darboux, l. c., pages 383, 386.

This equation determines ϕ as a function of v only, but as the curves $v =$ constant are not plane curves, ϕ is a function of both u and v, hence the equation must reduce to an identity. Equating to zero the coefficients of the various powers of cosec ϕ, we get

$$\frac{\tau_2}{\tau} = 0, \quad \frac{\tau_2}{\tau} - \frac{k_2}{k} = 0,$$

or $\rho =$ constant, $\tau =$ constant. Substituting in the preceding equations, we obtain finally $C = 0$. Hence no surfaces exist of the kind we seek.

§ 2

Let us now assume that $\rho/\tau = \alpha$, a function of v only, and that the curves $u =$ constant are geodesics. Having regard to the properties of geodesics, we may put $C = 1$.

The equations referred to above now become

$$p = 0, \quad q_1 = 0, \quad r_1 = 0,$$

$$\text{(i)} \quad r = \frac{\sin\phi}{\rho} A, \qquad \text{(i')} \quad r = -A_2,$$

$$\text{(ii)} \quad q = -\frac{\cos\phi}{\rho} A, \qquad \text{(iii)} \quad (r)_2 + qp_1 = 0,$$

$$\text{(iv)} \quad p_1 = \frac{1}{r}(q)_2, \qquad \text{(iv')} \quad p_1 = V.*$$

There are two cases to be considered according as V is zero or not. If V is not zero, take a new variable t so that $Vdv = dt$. From (iii), (iv)

$$\frac{\partial^2 r}{\partial t^2} + r = 0, \quad r^2 + q^2 = U'^2,$$

$$\therefore r = U' \sin(t + U), \quad q = -U'\cos(t + U).$$

From (i), (ii)

$$t + U = \phi, \quad -A_2 = U' \sin(t + U).$$

We can absorb U' into A, which is equivalent to reducing U' to unity, and we obtain finally

* Capital letters U, V denote functions of the variables u alone, v alone respectively.

$$A = \rho, \quad C = 1, \quad \rho_2 = -\sin(t + U), \quad U = \alpha u.$$

The last result shows that α, which is a function of v only, must be an absolute constant; also ρ must be of the form $a \cos \alpha u + b \sin \alpha u$, where a and b are functions of v alone and

$$a_2 = -\sin t, \quad b_2 = -\cos t.$$

All the conditions are now satisfied, so that surfaces do exist having the required properties, and to obtain them we have the equations

$$x_1^2 + y_1^2 + z_1^2 = \rho^2, \quad x_1x_2 + y_1y_2 + z_1z_2 = 0, \quad x_2^2 + y_2^2 + z_2^2 = 1.$$

$$D = \rho^2 \cos(t + ku), \quad D' = 0, \quad D'' = \rho V^*$$

These can be integrated and we obtain as coordinates of a point on a surface referred to a special set of axes

$$x = \rho_1 \frac{\sin(\sqrt{1 + \alpha^2}\,.\,u)}{\sqrt{1 + \alpha^2}} - \rho \cos \sqrt{(1 + \alpha^2}\,.\,u),$$

$$y = \rho_1 \frac{\cos(\sqrt{1 + \alpha^2}\,.\,u)}{\sqrt{1 + \alpha^2}} + \rho \sin(\sqrt{1 + \alpha^2}\,.\,u),$$

$$z = \frac{\rho_1}{\alpha\sqrt{1 + \alpha^2}}.$$

From a further consideration of the equation given on pages 383, 386, l. c., it appears that the curves $u =$ constant are plane curves, whose radii of curvature are given by

$$1/\rho = p_1 = V.$$

This shows that the set $u =$ constant consists of the different positions of an invariable curve in a plane which moves in a direction perpendicular to itself.

If $V = 0$, we see at once that the surface is a developable whose coordinates may be written in the form

$$x = a + lv, \quad y = b + mv, \quad z = c + nv,$$

where l, m, n, a, b, c are all functions of u only, and in virtue

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* Darboux, l. c., page 378.

of the special properties of the curves $u=$ constant, $v=$ constant, satisfy the following relations:

$$l^2+m^2+n^2=1, \quad l_1^2+m_1^2+n_1^2=1,$$

$$\frac{a_1}{l_1}=\frac{b_1}{m_1}=\frac{c_1}{n_1}=\beta, \text{ say.}$$

Then we have $A=v+\beta$ and $ds^2=(v+\beta)^2\,du^2+dv^2$. The equations to determine p, q, r, p_1, q_1, r_1, now become

$$p=0, \quad q_1=0, \quad r_1=0, \quad p_1=0,$$

$$(1) \quad r=-1, \qquad r=\frac{\sin\phi}{\rho}(v+\beta),$$

$$(2) \quad q=U, \qquad (2') \quad q=-\frac{\cos\phi}{\rho}(v+\beta),$$

$$(3) \quad \tau\phi_1=v+\beta.$$

Combining (1), (1′), (2), (2′) we obtain

$$\tan\phi=\frac{1}{U},$$

which, upon differentiation with respect to u, and substitution from (3), leads to the following equation:

$$\alpha=\frac{\rho}{\tau}=\frac{U_1}{(1+U^2)^{\frac{3}{2}}}.$$

From this it is apparent that ρ/τ, which is a function of v only, is an absolute constant. By integration we obtain

$$\frac{U}{\sqrt{1+U^2}}=\alpha u+\gamma \qquad (\gamma=\text{constant}).$$

No generality is lost by making $\gamma=0$, and then

$$U=\frac{\alpha u}{\sqrt{1-\alpha^2u^2}}.$$

The equations to be integrated are

$$\sum l^2 = 1, \quad \sum l_1^2 = 1, \quad \frac{l_1}{a_1} = \frac{m_1}{b_1} = \frac{n_1}{c_1} = \frac{1}{\beta},$$

$$\begin{vmatrix} l & m & n \\ l_1 & m_1 & n_1 \\ l_{11} & m_{11} & n_{11} \end{vmatrix} = U$$

and the surface is defined by the equations

$$x = a + kuv,$$

$$y = b + \frac{kuv}{U\sqrt{1-k^2}}(\cos\psi + k\sin\psi\, U),$$

$$z = c + \frac{kuv}{U\sqrt{1-k^2}}(\sin\psi - kU\cos\psi),$$

where a, b, c are perfectly determinate functions of u, and

$$k = \frac{\alpha}{\sqrt{1+\alpha^2}}, \quad \sin k\psi = \frac{ku}{\sqrt{1-k^2}}.$$

BRYN MAWR COLLEGE, PENNA.,
December, 1908.

The Differential Equations Satisfied by Abelian Theta Functions of Genus Three.

By J. Edmund Wright.

In several papers* and in his book "Multiply Periodic Functions,"† Baker has given the differential equations satisfied by hyperelliptic theta functions. His method is most satisfactory in its final outcome, because the constants that occur in the equations are expressed in terms of the associated Riemann Surface, with definitely known cross-cuts, but owing to this complete determination of the equations the process involves long and complicated algebraic manipulation. Its application to any but the hyperelliptic functions would seem almost impossible. Now if we start from the general definition of a theta function as a uniform integral function of several variables, that possesses certain period properties, we can discover enough about its nature to enable us to give the general forms of the differential equations it satisfies, and then it appears that conditions of coexistence of these equations are sufficient to make them precise.

For example, the general nature of the theta functions of genus 2 leads us to the conclusion that it must satisfy five differential equations of the fourth order, of a certain particular form. These equations involve twenty constants, but conditions to be satisfied in order that they may coexist reduces this number, so that finally the equations depend on only three essential constants, and therefore we conclude that the general solution of the final differential equations must be such a theta function. It is possible that, the equations once obtained, they may be integrated directly, and thus the theory may be made complete from this point of view.‡

The purpose of this paper is to determine the differential equations whose general solution is a general theta function of genus 3. As a first illustration,

* *Proc. Camb. Phil. Soc.*, Vol. IX (1898), p. 517; *ibid.*, Vol. XII (1903), p. 219; *Acta Math.*, t. XXVII (1903), p. 135.

† Cambridge University Press (1907).

‡ Cf. "*Multiply Periodic Functions*," p. 44 sqq.

we propose to apply the method outlined to the case of $p=2$ to obtain Baker's results.

The case of $p=3$ leads to a division of the types of equation into two classes. The first of these turns out to be the hyperelliptic case. By adding a suitable exponential factor to the theta function the equations are given by means of covariants of certain ternary forms; these forms are: 1) a quadratic whose coefficients are the second derivatives of the logarithm of the theta function; 2) a cubic whose coefficients are the third derivatives; 3) a quartic whose coefficients are the fourth derivatives, and similarly for higher derivatives, and 4) certain fixed forms. For the hyperelliptic case the fixed forms are a conic and a quartic. These two curves determine a binary octavic, namely that cut out on the conic by the quartic, and this case is thus associated with the invariant and covariant properties of a binary octavic. In the non-hyperelliptic case there is only one fixed form, a general quartic. It thus appears that this case is closely connected with the geometrical properties of a general quartic. This is interesting in view of the fact that a non-hyperelliptic curve of genus three can always be birationally transformed into a non-singular quartic, whereas this is not true of a hyperelliptic curve of genus three, for which the reduced curve of lowest order is a quintic with a triple point.

§ 1.

In the subsequent work we need some general definitions and theorems, which we quote from Baker's "Abelian Functions." *

Suppose that we have four matrices $\omega, \omega', \eta, \eta'$, each of p rows and columns, which satisfy the conditions: 1) that the determinant of ω is not zero; 2) that the matrix $\omega^{-1}\omega' (\equiv \tau)$ is symmetrical; 3) that for real values of $n_1, n_2, \ldots, n_p$ the quadratic form $\omega^{-1}\omega' n^2$ has its imaginary part positive; 4) that the matrix $\eta\omega^{-1}$ is symmetrical; 5) that $\eta' = \eta\omega^{-1}\omega' - \frac{1}{2}\pi i\bar{\omega}^{-1}$. We put

$$a = \tfrac{1}{2}\eta\omega^{-1}, \quad h = \tfrac{1}{2}\pi i\omega^{-1}, \quad b = \pi i\omega^{-1}\omega',$$

so that

$$\eta = 2a\omega, \quad \eta' = 2a\omega' - \bar{h}, \quad h\omega = \tfrac{1}{2}\pi i, \quad h\omega' = \tfrac{1}{2}b;$$

and we write

$$\lambda_m(u) = H_m(u + \tfrac{1}{2}\Omega_m) - \pi i m m',$$

where

$$H_m = 2\eta m + 2\eta' m', \quad \Omega_m = 2\omega m + 2\omega' m'.$$

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* Hereafter quoted as A. F.

Also let Q, Q' denote two assigned rows of p rational quantities, and suppose $\Pi(u)$ to be an integral function of the p arguments $u_1, u_2, \ldots, u_p$ that satisfies the equation

$$\Pi(u+\Omega_m) = e^{r\lambda_m(u)+2\pi i(mQ'-m'Q)}\Pi(u)$$

for all integral values of m, m'. Then the function $\Pi(u)$ is called a theta function of order r, with the associated constants 2ω, $2\omega'$, 2η, $2\eta'$, and the characteristic (Q, Q'). [A. F. 447, 448.]

It may be proved that the function $\Pi(u)$ exists, and further that if the associated constants and the characteristic are given, there are not more than r^p such functions linearly independent of one another. [A. F. 448–452.]

We notice that the essential character of $\Pi(u)$ is unchanged if a linear transformation be made on the variables u, or if it is multiplied by an exponential factor of the type e^{au^2}, where a is a symmetrical matrix.

The limit to the number of linearly independent functions of order r may be reduced if $\Pi(u)$ is an even function or an odd function of its arguments taken together. In this case it is not difficult to prove that the constants Q, Q' must be half integers [A. F. 462], and the results are:

If $\Pi(-u) = \varepsilon\Pi(u)$, where $\varepsilon = \pm 1$, and r is even, whilst (Q, Q') consists of integers, the number of linearly independent functions $\Pi(u)$ is

$$\leq \tfrac{1}{2}r^p + 2^{p-1}\varepsilon.$$

When r is odd, or when r is even and the characteristic (Q, Q') does not consist wholly of integers, then the number of linearly independent functions is

$$\leq \tfrac{1}{2}r^p + \tfrac{1}{4}[1-(-1)^r]\,\varepsilon e^{4\pi iQQ'}. \quad \text{[A. F. 463.]}$$

Now suppose θ to be a function of the first order, with half-integer characteristic (q, q'). Then from the properties of such functions we have the result that $e^{4\pi iqq'} = \varepsilon$. [A. F. 251.]

If $\Pi(u)$ is θ^r, it is clear that Π is of the r-th order, with characteristic (rq, rq'). Hence, for functions with the same defining properties as θ^r the above numbers become $\frac{1}{2}r^p + 2^{p-1}$ if r is even, and $\frac{1}{2}(r^p+1)$ if r is odd.

In particular we note that if $r=2$, $\theta(u+v)\theta(u-v)$ has the same defining properties as θ^2, and hence there must be a linear relation, with coefficients independent of u, connecting 2^p+1 of these functions for 2^p+1 different values of the arguments v. There is a similar result for functions of the type $\theta(u-v)\,\theta(u-w)\,\theta(u+v+w)$ when $r=3$, and so on for higher values of r.

Now if $F(u)$ is a function multiply periodic in ω, ω', *i. e.*, one such that

$$F(u+\Omega_m)=F(u),$$

for all values of the integers m, m', and if it is made integral by being multiplied by some power, s, of θ, it is clear that if $r>s$, $\theta^r F(u)$ has the same defining properties as θ^r, and hence there is a linear relation among either $\frac{1}{2}r^p+2^{p-1}$ or $\frac{1}{2}(r^p+1)$ such functions, according as r is even or odd, provided $F(u)$ is even.

The second derivatives of $\log\theta$ are such multiply periodic functions. BAKER, BOLZA and others use the notation $\wp_{ij}(u)=-\frac{\partial^2}{\partial u_i\,\partial u_j}(\log\theta)$. [See, *e. g.*, A. F. 292, etc.] We shall find it convenient to use (ij) for this function, and similarly, in general,

$$-\frac{\partial^k}{\partial u_r\,\partial u_s\,\partial u_t\ldots\ldots}(\log\theta)\text{ is written }(rst\ldots\ldots);$$

and $(rst\ldots\ldots)$ is a multiply periodic function, which is made integral on multiplication by θ^k.

We shall assume in the remainder of this paper that there is no polynomial relation of either the first or second order connecting the derivatives (ij). Such a relation would in fact be a limitation on the generality of the constants in ω, ω'. For example, if $p=2$, we could reduce a linear relation among (11), (12), (22) to the form $(12)=0$, and θ would reduce to the product of two elliptic theta functions. The particular cases for which such a relation exists are of some interest when $p=3$. We propose to consider them in a later paper.

§ 2.

We consider first the case $p=2$. In this case there are four linearly independent functions of the second order with the same period properties as θ^2. They are θ^2, $\theta^2(11)$, $\theta^2(12)$, $\theta^2(22)$. It is easy to verify that $[(11)(22)-(12)^2]\,\theta^3$ is integral, and hence the five functions with the same period properties as θ^3 are

$$\theta^3\{[(11)(22)-(12)^2],\ (11),\ (12),\ (22),\ 1\}.$$

To save repetition we shall say that a function $f(u)$ is of the r-th order when $f(u)$ is even, multiply periodic, and $\theta^r f(u)$ is integral. It is clear that a function of the r-th order is also a function of the s-th order if r is less than s.

The functions of the fourth order, ten in number, are

$$1,\ (11),\ (12),\ (22), \text{ and all products of the type } (pq)(rs).$$

Of the sixth-order functions twenty are linearly independent. Now they include I) $[(11)(22)-(12)]^2$, II) all products $(pq)(rs)(tu)$, III) the functions of the fourth order.

These are in number 21, and therefore they must be connected by a linear relation. We thus see *a priori* that there is a quartic relation among the three second derivatives (11), (12), (22). This turns out to be Kummer's Quartic Surface.

Again (pqr) is of the third order, except that it is an odd function, and hence any product $(pqr)(stu)$, being of the sixth order, must be expressible as a cubic polynomial in (11), (12), (22). These considerations are useful as showing the kind of relations we are to expect. To obtain them we make use of the fact that

$$\frac{\theta(u-v)\,\theta(u+v)}{\theta^2(u)}$$

is a function of the second order for all values of the variables v, and hence if it is expanded in powers of the v's, all its coefficients are such functions. We write $\theta = e^{-f}$; then

$$\frac{\theta(u-v)\,\theta(u+v)}{\theta^2(u)} = exp\left\{-2\left[\frac{1}{\lfloor 2}\left(v\frac{\partial}{\partial u}\right)^2 f + \frac{1}{\lfloor 4}\left(v\frac{\partial}{\partial u}\right)^4 f + \dots\right]\right\}.$$

If the right-hand side be expanded, the coefficients of products of the fourth and sixth orders of v are readily obtained, and we have

$$(pqrs) - 2[(pq)(rs) + (pr)(qs) + (ps)(qr)],$$
$$(pqrstu) - 2\Sigma(pq)(rstu) + 4\Sigma(pq)(rs)(tu),$$

for these coefficients, where the summations extend to all possible combinations of the six letters p, q, r, s, t, u. These expressions are therefore both of them functions of the second order.

Hence we must have $(pqrs) - 2[(pq)(rs) + (pr)(qs) + (ps)(qr)] =$ a linear function of 1, (11), (12), (22), $= \sum_{h,k} b_{pqrs}^{(hk)}(hk) + b_{pqrs}$, say, where the b's are constants, and the summation extends once to each pair of values of h, k. Now, by giving p, q, r, s the values 1, 2 we obtain five such equations. These are differential equations of the fourth order for a single function f, and their coexistence by no means follows for general values of the constants b.

In fact, if we differentiate the five equations once, and eliminate fifth derivatives, we obtain four equations among third and second derivatives; these are homogeneous and linear in the third derivatives, which may therefore be eliminated. We thus have an equation among second derivatives only. We might use this equation to obtain by differentiation other homogeneous linear equations in third derivatives, and then by elimination other equations connecting the second derivatives. It is simpler, however, to differentiate the five fundamental differential equations twice and then to eliminate from them the sixth derivatives that occur. We thus get eight equations involving second derivatives and certain functions of third derivatives of the type $(pqr)(hks)-(pqs)(hkr)$. There are only three of these latter functions, and thus by elimination we get five equations which turn out to be of the form

$$A_i[(11)(22)-(12)^2]+B_i(11)+C_i(12)+D_i(22)+E_i=0 \quad (i=1, 2, \ldots, 5),$$

where A, B, C, D, E are constants. As we have assumed that no such relation exists, A, B, C, D, E must all be zero.

If we denote the right-hand side of the typical fundamental equation by B_{pqrs}, and additional suffixes denote differentiations, and if $B_{pqra,\,s\beta}-6(s\beta)B_{pqra}$, after fourth derivatives have been replaced by their values in terms of second derivatives, is written $[pqra,\, s\beta]$, it is not difficult to see that a typical one of the equations just mentioned is

$$\begin{aligned}[pqra,\, s\beta]+[pqas,\, r\beta]+[pars,\, q\beta]+[aqrs,\, p\beta]\\ =[pqr\beta,\, sa]+[pq\beta s,\, ra]+[p\beta rs,\, qa]+[\beta qrs,\, pa].\end{aligned}$$

The five equations are therefore

$$\begin{gathered}[1111,\, 12]=[1112,\, 11], \quad [1112,\, 22]=[1222,\, 11],\\ [1111,\, 22]+2[1112,\, 12]=3[1122,\, 11],\end{gathered}$$

and two similar equations obtained by interchanging 1 and 2. We take the second-degree terms in these equations first. $[1111, 12]=[1112, 11]$ becomes, on expansion,

$$(2b_{1111}^{(12)}+4b_{1112}^{(22)})[(11)(22)-(12)^2]+\text{linear terms}=0.$$

Thus we have $b_{1111}^{(12)}+2b_{1112}^{(22)}=0$.

Similarly, from the second equation we have $b_{1112}^{(11)}=b_{1222}^{(22)}$, and from the third $b_{1111}^{(11)}-b_{1112}^{(12)}=3b_{1122}^{(22)}$.

We thus have five relations among the constants b. If these are satisfied, it appears that the remaining relations obtained from the above five equations determine the values of the constants b_{pqrs} uniquely, and otherwise lead to no new relations.

Now, the multiplication of θ by an exponential factor ϵ^{au^2} does not alter its essential nature, and is equivalent to giving arbitrary additive constants to the derivatives (11), (12), (22). If we thus modify our equations, we may make

$$2b^{(11)}_{1111} + b^{(12)}_{1112} = 0, \quad 2b^{(22)}_{2222} + b^{(12)}_{1222} = 0, \quad 2b^{(11)}_{1112} + b^{(22)}_{1122} = 0,$$

and then the fundamental equations take precisely the form given by Baker, "Multiply Periodic Functions," p. 49.

We can now verify without trouble that the equations are compatible, obtain the Kummer and Weddle Surfaces, and obtain the relations for products of third derivatives, exactly as in Baker. We are, however, using this case as an illustration of the method for $p = 3$, and hence shall indicate in outline another method for getting $(pqr)(stu)$ and the Kummer Surface.

We use Baker's notation for the coefficients of the differential equations for fourth derivatives, so that

$$\begin{aligned}
-\tfrac{1}{3}B_{1111} &= a_0a_4 - 4a_1a_3 + 3a_2^2 + a_2(11) - 2a_1(12) + a_0(22),\\
-\tfrac{1}{3}B_{1112} &= \tfrac{1}{2}(a_0a_5 - 3a_1a_4 + 2a_2a_3) + a_3(11) - 2a_2(12) + a_1(22),\\
-\tfrac{1}{3}B_{1122} &= \tfrac{1}{6}(a_0a_6 - 9a_2a_4 + 8a_3^2) + a_4(11) - 2a_3(12) + a_2(22),\\
-\tfrac{1}{3}B_{1222} &= \tfrac{1}{2}(a_1a_6 - 3a_2a_5 + 2a_3a_4) + a_5(11) - 2a_4(12) + a_3(22),\\
-\tfrac{1}{3}B_{2222} &= a_2a_6 - 4a_3a_5 + 3a_4^2 + a_6(11) - 2a_5(12) + a_4(22).
\end{aligned}$$

It is interesting to notice the covariantive form of the equations. If we take du_1 and du_2 as variables x_1 and x_2, then any linear transformation on the u's is equivalent to the same linear transformation on the x's; it is clear that $d^4\wp$ and $d^2\wp$,

$$\equiv (11)x_1^2 + 2(12)x_1x_2 + (22)x_2^2, \quad \equiv a_x^2 \equiv b_x^2 \equiv c_x^2 \equiv \ldots,$$

are invariant under such a transformation. Also

$$(a_0, a_1, a_2, \ldots, a_6)(x_1x_2)^6 \equiv \alpha_x^6 \equiv \beta_x^6 \equiv \ldots$$

is an associated sextic, and our fundamental equations are given by

$$d^4\wp - 6(d^2\wp)^2 = -3(\alpha a)^2\alpha_x^4 - \tfrac{3}{2}(\alpha\beta)^4\alpha_x^2\beta_x^2, \equiv B, \text{ say.}$$

From the coefficient, already given, of v^6 in the original expansion we have $d^6\wp - 30d^2\wp\, d^4\wp + 60(d^2\wp)^3 =$ a homogeneous sextic in du_1, du_2, of which the

coefficients are functions of the second order. If we put this equal to X, X must have its coefficients linear functions of the second derivatives. Now $d^4\wp - 6(d^2\wp)^2 = B$, and hence on differentiation $d^6\wp - 12d^2\wp d^4\wp - 12(d^2\wp)^2 = d^2B$.

If we eliminate $d^6\wp$ and $d^4\wp$ from these equations, we have

$$12\{(d^3\wp)^2 - 4(d^2\wp)^3\} + d^2B - 18Bd^2\wp = X.$$

Now we have already indicated the method of obtaining expressions $(pqr)(hks) - (pqs)(hkr)$, and therefore we can obtain the various products $(pqr)(hks)$ by equating cofficients of powers and products of du_1, du_2, in the above equation, as soon as we know the coefficients of X. For example,

$$\begin{aligned}(111)^2 - 4(11)^3 + 3[a_2(11) - 2a_1(12) + a_0(22)](11) \\ + a_0[(11)(22) - (12)^2] = \lambda(11) + \mu(12) + \nu(22) + \rho,\end{aligned}$$

where λ, μ, ν, ρ are constants to be determined.

There are two methods for determining the unknown constants. In the first method we differentiate twice and eliminate fourth and fifth derivatives by means of the fundamental fourth-order equations. We also substitute for the various squares and products of third derivatives that occur. We then remain with an equation which turns out to be in some cases of the second degree, in others of the fourth degree in second derivatives. Now those of the second degree must vanish identically, and by equating their coefficients to zero we have enough equations to determine the unknown constants. If these are substituted in the fourth-degree equations obtained, they all reduce to one and the same equation, which is that of Kummer's Quartic Surface.

In the second method we differentiate the fundamental equations, and by subtraction eliminate fifth derivatives. We thus remain with an equation linear in third derivatives. Such an equation, for example, is

$$6(12)(111) - 6(11)(112) = 3a_3(111) - 9a_2(112) + 9a_1(122) - 3a_0(222).$$

We multiply these equations by the various third derivatives, and substitute for the squares and products of third derivatives their expressions in terms of second derivatives. As before, we get equations either of the second or of the fourth degree in second derivatives, and again we can determine the unknown constants and obtain Kummer's Quartic. In either case the work may be somewhat simplified, if we notice that $d^2B - 6Bd^2\wp = 4\alpha_x^6[(11)(22) - (12)^2]$ + a quantity

linear in second derivatives, by incorporating this linear quantity into X. The final result in symbolic form is

$$(d^3\wp)^2 - 4(d^2\wp)^3 = -\tfrac{1}{2}(ab)^2\alpha_x^6 - 3(\alpha a)^2\alpha_x^4 b_x^2 - 9(\alpha a)^2(\alpha\beta)^2\alpha_x^2\beta_x^4 + \tfrac{9}{2}(\alpha a)(\beta a)\alpha_x^3\beta_x^3 + \tfrac{27}{8}(\alpha\beta)^2(\beta\gamma)^2(\gamma\alpha)^2\alpha_x^2\beta_x^2\gamma_x^2.$$

The Kummer Quartic is an invariant of α_x^6 and a_x^2. Its symbolic expression is

$$4(ab)^2(cd)^2 - 16(\alpha a)^2(ab)^2(\alpha c)^2 + 6(\alpha a)^2(\beta b)^2(\alpha\beta)^4 + (ab)^2(\alpha\beta)^6 + 9(\alpha a)^2(\alpha\beta)^2(\alpha\gamma)^2(\beta\gamma)^4 - \tfrac{9}{8}(\beta\gamma)^2(\gamma\alpha)^2(\alpha\beta)^2(\alpha\delta)^2(\beta\delta)^2(\gamma\delta)^2 = 0.$$

If we write $d^3\wp, \equiv (111)x_1^3 + \ldots, \equiv p_x^3 \equiv q_x^3 \equiv r_x^3 \equiv \ldots,$ we have for the symbolic equation of the Weddle Surface

$$(hp)^2(hq)(pq)(qr)^2 = 0 \;\; [\text{where } h_x^3 \equiv (\alpha p)^3\alpha_x^3].$$

In the above work we have assumed that there exists no quadratic relation among (11), (12), (22); it is interesting to note that the assumption of the existence of such a relation leads to a linear relation among these quantities. By a proper choice of variables such a linear relation could be reduced either to $(11) = 0$, or to $(12) = 0$.

The latter shows that θ must be the product of two elliptic θ functions, whilst the former implies that θ is the product of an elliptic θ and an exponential, e^{Au_1+B}, where A and B are constants.

§ 3.

We now consider the case of $p = 3$. We shall find it convenient to use Δ to denote the determinant

$$\begin{vmatrix} (11), & (12), & (13) \\ (21), & (22), & (23) \\ (31), & (32), & (33) \end{vmatrix}$$

and Δ_{rs} to denote the cofactor of (rs) in Δ.

It may be verified without difficulty that Δ, though apparently of the sixth, is really of the fourth order, whilst Δ_{rs} is of the third order.

In this case there are eight functions of the second order. Of these we have seven, the quantities 1, (11), (12), (13), (22), (23), (33). There must be one linearly independent of these, and this one we call Y. It is to be noticed that for simplifying our equations we may modify Y by adding to it any linear function of the known second order functions (pq) and 1.

The number of linearly independent third-order functions is $\frac{1}{2}(3^3+1),=14$. We have the eight already given, and the six functions Δ_{rs}. We see at once that there are two cases according as these are or are not linearly independent. As we are assuming that no quadratic relation exists among the quantities (pq), we see that if these 14 third-order functions are not independent, there is a relation

$$Y = \Sigma a_{pq}\Delta_{pq} + \Sigma b_{pq}(pq) + c,$$

where the a's, b's and c are constants. This turns out to be the hyperelliptic case.

There are $\frac{1}{2}4^3 + 4 = 36$ functions of the fourth order. Now we have already eight of these, namely the second-order functions. In addition we must have all products $(pq)(rs)$, $(pq)Y$, Y^2, and Δ. These are in all 37, and hence they must be connected by at least one linear relation. In the hyperelliptic case this relation is the one given above. In the other we shall show later that it may be reduced to the form

$$Y^2 + 2\Delta = \text{a quadratic in the second derivatives } (pq).$$

The number of sixth-order functions is $\frac{1}{2}6^3 + 4 = 112$. We have

I)	21 products	$\Delta_{pq}\Delta_{rs}$,
II)	56 "	$(pq)(rs)(tu)$,
III)	21 "	$Y(pq)(rs)$,
IV)	6 "	$Y^2(pq)$,
V)	21 "	$(pq)(rs)$,
VI)	6 "	$Y(pq)$,
VII)	the 11 functions	(pq), $Y\Delta$, Y^3, Y^2, Y, 1,

that is to say, 142 such functions. These must therefore be connected by 30 linear relations.

In the hyperelliptic case we may neglect 6 of III) which reduce to combinations of I), II) and V); and if we limit ourselves to functions of the fourth or less degree in the derivatives we may neglect IV), $Y\Delta$, Y^3. Also we may neglect VI), Y^2, Y. We thus have 120 functions of degree not greater than four in the second derivatives. They must therefore be connected by at least eight linear relations. In fact it will appear later that there are fifteen such

linearly independent relations. (These relations must of course be connected. In fact, if we use the second derivatives as coordinates in space of six dimensions, we have eight five-folds which pass through a common three-fold. This three-fold is of the eighth order.)

In the non-hyperelliptic case we may neglect IV), which may be expressed in terms of I), II), etc., by the fourth-order relation, and similarly we may neglect Y^3 and Y^2. We thus have 134 functions, and they are at most linear in Y. There must thus be 22 relations of the type $YQ_i + A_i\Delta = K_i$, where Q_i is quadratic, K_i quartic in the second derivatives and A_i is a constant. By consideration of the functions of the fifth order we can show that six of these relations must be of the form $YQ_i = C_i$, where C_i is a cubic, and the quadratic Q_i is linear in the quantities Δ_{pq}, (pq).

Again, it may easily be shown that R_{pq}, $\equiv Y_{pq} - 6(pq)Y$, is of the third order, and therefore, if the case is not hyperelliptic, R_{pq} = a linear function of Δ_{pq}, (pq).

In the hyperelliptic case it appears that this is not true; in fact R_{pq} = a linear function of Δ_{pq}, (pq), J; where J is a certain function cubic in the derivatives, and J occurs in at least one of the expressions.

We now determine the equations in detail. In the first place

$$(pqrs) - 2[(pq)(rs) + (pr)(qs) + (ps)(qr)]$$

is of the second order, and is therefore a linear function of 1, (pq), Y. We thus have fifteen equations of the type

$$(pqrs) - 2[(pq)(rs) + (pr)(qs) + (ps)(qr)] = B_{pqrs} + a_{pqrs}Y, \qquad [p, q, r, s = 1, 2, 3], \tag{1}$$

where $B_{pqrs} = \Sigma\, b_{pqrs}^{(ij)}(ij) + b_{pqrs}$, the summation being taken once for each pair of values i, j, and the a's and b's are constants.

If this be differentiated with respect to the variables with suffixes t and u, we obtain

$$\begin{aligned}(pqrstu) - 2\{&(pqtu)(rs) + (pq)(rstu) + (prtu)(qs) + (pstu)(qr)\\ &+ (pr)(qstu) + (ps)(qrtu) + (pqt)(rsu) + (pqu)(rst) + (prt)(qsu)\\ &+ (pru)(qst) + (pst)(qru) + (psu)(qrt)\} = B_{pqrs,\,tu} + a_{pqrs}Y_{tu}.\end{aligned}$$

If s, t are interchanged we obtain another such equation. From these two the sixth derivative may be eliminated by subtraction. The result is

$$2\{(pqu, r)_{st} + (pru, q)_{st} + (qru, p)_{st} + (pq, ru)_{st} + (pr, qu)_{st} + (qr, pu)_{st}\}$$
$$= B_{pqrs,\,tu} - B_{pqrt,\,su} + a_{pqrs} Y_{tu} - a_{pqrt} Y_{su},$$

where

$$(pqu, r)_{st} \equiv (pqus)(rt) - (pqut)(rs),$$
$$(pq, ru)_{st} \equiv (pqs)(rut) - (pqt)(rus).$$

If we permute the suffixes $p, q, r, u,$ we obtain four such equations. They involve the three third-derivative expressions $(pq, ru)_{st}$ and certain fourth and second derivatives. By adding we eliminate the third derivatives, and then by means of (1) we may express fourth derivatives in terms of second derivatives and Y. We thus obtain the equation

$$[pqrs, ut] - [pqrt, us] + a_{pqrs} R_{ut} - a_{pqrt} R_{us} + \text{three similar expressions obtained by interchanging } u \text{ with } p \text{ and with } q \text{ and with } r = 0, \tag{2}$$

where $[pqrs, ut] = B_{pqrs,\,ut} - 6(ut) B_{pqrs}$, in which the fourth derivatives have been replaced by their values from (1).

In addition the four equations mentioned above serve to determine the quantities $(qr, pu)_{st}$. We have in fact

$$\begin{aligned}
&4(qr, pu)_{st} + 4\{[(uq)(sp) + (us)(pq)](rt) + [(ur)(sp) + (us)(rp)](qt) \\
&\quad - [(uq)(tp) + (ut)(pq)](rs) - [(ur)(tp) + (ut)(rp)](qs)\} + 4Y\{(rt)a_{pqsu} \\
&\quad - (rs)a_{pqtu} + (qt)a_{prsu} - (qs)a_{prtu} - (pt)a_{qrsu} + (ps)a_{qrtu} - (ut)a_{pqrs} \\
&\quad + (us)a_{pqrt}\} + 4\{(rt)B_{pqsu} - (rs)B_{pqtu} + (qt)B_{prsu} - (qs)B_{prtu} \\
&\quad - (pt)B_{qrsu} + (ps)B_{qrtu} - (ut)B_{pqrs} + (us)B_{pqrt}\} \\
&= a_{pqrs}R_{tu} - a_{pqrt}R_{su} + a_{qrsu}R_{tp} - a_{qrtu}R_{sp} \\
&\qquad + [pqrs, tu] - [pqrt, su] + [qrsu, tp] - [qrtu, sp].
\end{aligned} \tag{3}$$

It is worthy of remark that the above equations (2), (3) are of the same general form for any number of variables, for any value of p, the only difference being that if p is greater than three there is more than one function Y.

The equations (2) are linear in the quantities $R_{\alpha\beta}$, $\Delta_{\alpha\beta}$, $(\alpha\beta)$, Y. If we regard $R_{\alpha\beta}$ and Y as unknowns, there are different cases according as they can be solved for some or all of these variables or not. Suppose first that they can be solved for Y; then they give Y as a linear function of Δ_{pq}, (pq). By making

a convenient linear transformation on the fundamental variables this may be modified into one of the forms

$$Y=\Delta_{11}+\Delta_{22}+\Delta_{33}, \quad Y=\Delta_{23}, \quad Y=\Delta_{11}.$$

In this paper we neglect the second two forms, which lead to less general cases than the first, and assume that

$$Y=\Delta_{11}+\Delta_{22}+\Delta_{33}.$$

If we calculate R_{pq} directly from this value of Y, and then use (1), (3) to eliminate third and fourth derivatives, we obtain equations which show that R_{pp} can not be a linear function of the quantities Δ_{rs}, (rs), and from them we deduce without difficulty that

$$a_{pppp}=3, \quad a_{ppqq}=1, \quad a_{pppq}=0, \quad a_{ppqr}=0, \qquad (p \neq q \neq r).$$

If we multiply the 15 equations (1) by du_1^4, etc., and add, we obtain the equation

$$d^4\wp-6(d^2\wp)^2=B+3C^2Y,$$

where B is a homogeneous quartic in the differentials du, with coefficients linear in the second derivatives (pq), and C is $du_1^2+du_2^2+du_3^2$.

When the particular values given above for the a's are substituted in the equations (2), it appears that they may be solved for the five magnitudes

$$R_{11}-R_{22}, \quad R_{11}-R_{33}, \quad R_{23}, \quad R_{31}, \quad R_{12}.$$

Accordingly each of these five must be expressible as a linear function of the quantities Δ_{pq}, (pq). We therefore assume linear functions of this type, with unknown coefficients, for these magnitudes and substitute their values and that of Y in the equations (2). These equations are now quadratics in second derivatives only, and hence they must vanish identically. Hence by equating their coefficients to zero we obtain equations, which are in fact all that exist among the undetermined constants.

Now the complete determination of the coefficients would be somewhat long if we introduced no further restriction on our choice of variables u_1, u_2, u_3. We notice, however, that Y is an algebraic invariant of the ternary forms $d^2\wp$ and C. In fact, if $d^2\wp=a_x^2=b_x^2=\ldots$, and $C=A_x^2=B_x^2=C_x^2=\ldots$, where x is du, we have the relation $3Y(ABC)^2=(Aab)^2$.

There is thus a certain amount of freedom at our disposal which may be used to get the equations in canonical form. We may, for example, perform any linear transformation on the variables u_1, u_2, u_3, that leaves the quadric C invariant. Also we may add constants to the second derivatives (pq), provided we subtract an appropriate linear function of the second derivatives from Y. Further, it is suggested that the equations can be modified so as to be covariants of certain ternary forms. It appears that the constants to be added to (pq) are uniquely determinate if the equations are to be covariant, and hence that the covariant canonical form is perfectly definite.

The details of the process I adopted are as follows: I first wrote out the general equations among the constants, and then proved that by a certain transformation on the u's the coefficients of type $b_{1112}^{(33)}$ and certain constants in the R's could all be made zero. When this simplification was introduced, the constants were all readily calculated in terms of six left arbitrary, and the equations were then in a canonical, though not covariant shape. It was clear, however, that by slight modification they might be made covariantive if a certain fixed quartic were associated, and the proper modification was given without much trouble by comparing terms in B that involved second derivatives with possible covariants.

The fixed forms entering into the covariant equation are the conic C and a certain quartic. The quartic is trinodal, and its nodes are at the vertices of a self-conjugate triangle of the conic. It is clear that by taking C in the form $x_1x_3 - x_2^2$, introducing a parametric pair t_1, t_2, and writing $x_1 = t_1^2$, $x_2 = t_1t_2$, $x_3 = t_2^2$, we may make use of binariants involving a single octavic, that cut out on the conic by the quartic. This leads to a set of equations equivalent to the one given by Baker in the papers already quoted, and serves to identify our equations with the hyperelliptic case, for which a binary octavic is fundamental. I prefer, however, to keep C in the form $x_1^2 + x_2^2 + x_3^2$, so as to preserve symmetry, and to work with orthogonal invariants of a quartic. The quartic is taken to be

$$6\Sigma h_1 x_2^2 x_3^2 + 12\Sigma p_1 x_1^2 x_2 x_3 \equiv \alpha_x^4 \equiv \beta_x^4 \equiv \gamma_x^4 \equiv \dots .$$

It involves only five constants, namely the ratios of the six quantities h_1, h_2, h_3, p_1, p_2, p_3. This was to be expected, since the hyperelliptic functions for $p = 3$ possess only five class-moduli. In the non-hyperelliptic case we expect six essential constants.

The values of the coefficients of B for the canonical form are the following:

$$B_{1111}=6(h_2+h_3)(11)+6h_2(22)+6h_3(33)-12p_1(23)+b_{1111},$$
$$B_{1112}=3p_3[(11)+(22)]-6p_2(23)+b_{1112},$$
$$B_{1122}=(h_1+h_2-h_3)[(11)+(22)]-4h_3(33)+2p_1(23)+2p_2(13)+b_{1122},$$
$$B_{1123}=-p_1[2(11)+3(22)+3(33)]-2(h_2+h_3-h_1)(23)$$
$$+2p_2(12)+2p_3(13)+b_{1123},$$

where

$$b_{1111}=3(L+M)-6p_1^2,$$
$$b_{1112}=3p_3(h_2+h_3-h_1)-3p_1p_2,$$
$$b_{1122}=(L-M)-(h_3^2+2h_1h_2-h_1^2-h_2^2)-2p_3^2,$$
$$b_{1123}=-2p_1(2h_1+h_2+h_3)+p_2p_3,$$

and

$$4L=2(h_2h_3+h_3h_1+h_1h_2)-h_1^2-h_2^2-h_3^2,$$
$$M=p_1^2+p_2^2+p_3^2,$$

and the remainder of the B's are obtained by interchanging the suffixes 1, 2, 3.

We can identify the equations with an appropriate symbolic expression by making use of the fact that orthogonal invariants consist of sums of products of symbolic expressions of the types

$$(\alpha_1^2+\alpha_2^2+\alpha_3^2),\quad (\alpha_1\beta_1+\alpha_2\beta_2+\alpha_3\beta_3),\quad \alpha_x,\quad (\alpha\beta\gamma),$$

and then introducing the fundamental conic to obtain the regular ternary three-rowed determinants. We thus obtain the result

$$d^4\wp-6(d^2\wp)^2=6(A\alpha a)^2\alpha_x^2B_x^2+4(AB\alpha)(ABa)\alpha_x^3a_x-3(ABa)^2\alpha_x^4+3YA_x^2B_x^2+Q,$$

where

$$Q=-\tfrac{1}{8}(A\alpha\beta)^2(B\alpha\beta)^2C_x^2D_x^2+2(A\alpha\beta)^2(B\alpha\beta)(BC\beta)C_xD_x^2\alpha_x-\tfrac{1}{2}(AB\beta)^2(CD\beta)^2\alpha_x^4$$
$$-\tfrac{1}{4}(AB\alpha)^2(CD\beta)^2\alpha_x^2\beta_x^2-\tfrac{5}{4}(AB\alpha)(AB\beta)(CD\alpha)(CD\beta)\alpha_x^2\beta_x^2$$
$$+2(AB\beta)^2(CD\alpha)(CD\beta)\alpha_x^3\beta_x.$$

In the course of determining the above equations we also determine the values of R_{23}, R_{31}, R_{12}, $R_{11}-R_{22}$, $R_{11}-R_{33}$. By direct differentiation of Y we can calculate, for example, R_{11}, and thus we have expressions for the quantities $R_{pq}\equiv Y_{pq}-6(pq)Y$ in terms of second derivatives. Now R_{11}, for example, involves the as yet undetermined third-order function J. Let

$$I=\tfrac{1}{2}(A\alpha a)^2(B\alpha b)^2,=\Sigma h_1(11)^2+(h_1+h_2+h_3)\Sigma(11)(22)$$
$$-2\Sigma p_1(23)[(22)+(33)]+4\Sigma p_1(12)(13),$$

then

$$J=2Y[(11)+(22)+(33)]-\Delta+I,$$

and

$$
\begin{aligned}
R_{11} = & -4J - 4(h_1 + h_2 + h_3)(\Delta_{22} + \Delta_{33}) - 8h_1\Delta_{11} + 8p_2\Delta_{31} + 8p_3\Delta_{12} \\
& + [2p_1^2 + 8p_2^2 + 8p_3^2 - 14L + 3(h_1^2 + 2h_2h_3 - h_2^2 - h_3^2)](11) \\
& + [8p_1^2 + 4p_2^2 + 14p_3^2 - 8L + (h_2^2 + 2h_3h_1 - h_3^2 - h_1^2)](22) \\
& + [8p_1^2 + 14p_2^2 + 4p_3^2 - 8L + (h_3^2 + 2h_1h_2 - h_1^2 - h_2^2)](33) \\
& - [22p_2p_3 - 4p_1(4h_1 - h_2 - h_3)](23) - [14p_3p_1 + 2p_2(7h_1 - h_2 + h_3)](31) \\
& - [14p_1p_2 + 2p_3(7h_1 + h_2 - h_3)](12) - (h_2 + h_3 - h_1)(h_3 + h_1 - h_2)(h_1 + h_2 - h_3) \\
& + 2p_1^2(7h_1 + 2h_2 + 2h_3) + 2p_2^2(2h_1 + 7h_2 + 2h_3) + 2p_3^2(2h_1 + 2h_2 + 7h_3) \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - 16p_1p_2p_3 .
\end{aligned}
$$

Also

$$
\begin{aligned}
R_{12} = & \; 4p_3(\Delta_{11} + \Delta_{22}) - 8p_1\Delta_{13} - 8p_2\Delta_{23} - 8h_3\Delta_{12} \\
& + [p_3(h_2 - 5h_1 - h_3) - 2p_1p_2](11) + [p_3(h_1 - 5h_2 - h_3) - 2p_1p_2](22) \\
& + [6p_3h_3 - 6p_1p_2](33) + [4(h_1 + 2h_2 + h_3)p_2 - 2p_1p_3](23) \\
& + [4(2h_1 + h_2 + h_3)p_1 - 2p_2p_3](31) \\
& + [6(L + M) - 4p_1^2 - 4p_2^2 + 4h_3(h_3 - h_1 - h_2)](12).
\end{aligned}
$$

These expressions must of course be coefficients of covariants. The determination of the appropriate symbolic expressions involves rather long calculation for the lower-degree terms in second derivatives. We have however the result that

$$
\begin{aligned}
\tfrac{1}{6}(ABC)^2[d^2Y - 6Y(d^2\varphi)] = C\{-4J + (\alpha ab)^2(\alpha AB)^2 - (\alpha AB)^2(\alpha CD)^2 Y\} \\
+ 2(Aab)(Bab)(AC\alpha)(BC\alpha)\alpha_x^2 + H + KC,
\end{aligned}
$$

where H is a quadratic covariant linear in the quantities (pq), and K is an invariant with constant coefficients.

We next need the expressions, which we know *a priori* to exist, for products of third derivatives as cubic functions of second derivatives. It seems difficult to find these by direct integration, as may be done for $p = 2$, because the complication introduced by the function Y makes the algebra heavy. In the case of $p = 2$ the cubics are minors of a four-rowed determinant. The corresponding determinantal expressions in our case are not at all obvious, for instead of a four-rowed determinant we now have an array with ten rows and twenty-four columns, all of whose ten-rowed determinants must vanish. The third derivatives are proportional to first minors of these determinants, but it would seem impossible to readily factor the nine-rowed determinants, since there are many quartic relations among their elements.

We therefore use the method indicated for $p=2$. We have, as before, the equation

$$d^6\wp - 30\,d^2\wp\, d^4\wp + 60\,(d^2\wp)^3 = 12X,$$

where X is a covariant of the sixth order, with coefficients of the first degree in (pq), Y. Also

$$d^4\wp - 6\,(d^2\wp)^2 = B + 3\,YC^2.$$

Hence, by differentiation

$$d^6\wp - 12\,(d^2\wp)(d^4\wp) - 12\,(d^3\wp)^2 = d^2B - 3\,C^2 d^2 Y,$$

and therefore, by elimination of $d^6\wp$, $d^4\wp$, we have the equation

$$\begin{aligned}(d^3\wp)^2 - 4\,(d^2\wp)^3 - (B + 3\,YC^2)\,d^2\wp + \tfrac{1}{12}[d^2B - 6\,B(d^2\wp)]&\\ + \tfrac{1}{4}C^2[d^2Y - 6\,Y(d^2\wp)] = X.&\qquad (4)\end{aligned}$$

The terms of the second degree in $d^2B - 6B(d^2\wp)$ may be readily obtained from the symbolic expression for B. They are all linear functions of the quantities Δ_{pq}, (pq), and hence are only of the third order. Also the quantities $d^2Y - 6\,Y(d^2\wp)$ are known, so that the coefficients of the left-hand side of (4) are easily obtained. We can therefore express such functions as (111), (111)(112), $2(111)(122) + 3(112)^2$, etc., by means of quantities of the third degree in second derivatives, and all the coefficients except those arising from X are known. Again, (3) gives, for example, $(111)(122) - (112)^2$ as a third-degree expression, and hence we can find (111)(122) and $(112)^2$. Similarly we can find expressions for all the remaining products of third derivatives, though the coefficients arising from X are as yet undetermined. Now $Y = \Delta_{11} + \Delta_{22} + \Delta_{33}$ and therefore

$$\begin{aligned}Y_1 = (111)[(22) + (33)] + (122)[(33) + (11)] + (133)[(11) + (22)]&\\ - 2(123)(23) - 2(113)(13) - 2(112)(12).&\end{aligned}$$

We multiply this equation by any third derivative, and substitute the expressions already obtained for products of third derivatives. We thus obtain the values of such products as $(pqr)Y_s$ in terms of second derivatives, and in these expressions all the coefficients are known except those of the second and lower degrees and those associated with such functions as $(pq)Y$.

We now obtain, by differentiation and subtraction of the fundamental equations (1), 24 equations linear in second derivatives, linear in third derivatives, and possibly containing a term that is a first derivative of Y.

For example, we have

$$\begin{aligned}2(111)(12) - 2(112)(11) = 2\,(h_2 + h_3)(112) + 2\,h_2(222) + 2\,h_3(233)&\\ - 4p_1(223) - p_3[(111) + (122)] + 2\,p_2(123) + Y_1.&\end{aligned}$$

We multiply these equations by one of the third derivatives, substitute for products of third derivatives, and obtain finally an equation which is either of the fourth or of the second degree in second derivatives. The fact that the equations of the second degree must vanish identically enables us to determine the unknown constants, and then the remaining equations give the relations of the fourth degree among the second derivatives.

The work indicated above is very long, both on account of the number of functions to be calculated, and on account of the magnitude of the final results. We content ourselves therefore, for the present, with giving the terms of highest degree for the various functions mentioned. The typical products of third derivatives are given by the equations

$$
\begin{aligned}
&(111)^2 - 4(11)^3 - J - 3(11)Y = \ldots\ldots, \\
&(111)(112) - 4(11)^2(12) - (12)Y = \ldots\ldots, \\
&(111)(122) - 2(11)^2(22) - 2(11)(12)^2 + J - [(11) + 2(22)]\,Y = \ldots\ldots, \\
&(112)^2 - 4(11)(12)^2 + Y(22) - J = \ldots\ldots, \\
&(111)(123) - 2(11)(12)(13) - 2(11)^2(23) - 2Y(23) = \ldots\ldots, \\
&(112)(113) - 4(11)(12)(13) + Y(23) = \ldots\ldots, \\
&(111)(222) + 2(12)^3 - 6(11)(22)(12) + 3Y(12) = \ldots\ldots, \\
&(112)(122) - 2(11)(22)(12) - 2(12)^3 - Y(12) = \ldots\ldots, \\
&(111)(223) - 4(11)(12)(23) - 2(11)(13)(22) + 2(12)^2(13) + Y(13) = \ldots\ldots, \\
&(112)(123) - 2(11)(12)(23) - 2(12)^2(13) = \ldots\ldots, \\
&(113)(122) - 2(11)(13)(22) - 2(12)^2(13) - Y(13) = \ldots\ldots, \\
&(123)^2 - 4(23)(31)(12) + \Delta - Y[(11) + (22) + (33)] = \ldots\ldots, \\
&(112)(233) - 2(11)(23)^2 - 2(33)(12)^2 - \Delta + Y(22) = \ldots\ldots,
\end{aligned}
$$

where the terms omitted are of the second or lower degree in the second derivatives.

Those for the products $(pqr)\,Y_s$ are

$$
\begin{aligned}
(111)\,Y_1 &= Y[4(11)^2 + 4\Delta_{11} + 2\Delta_{22} + 2\Delta_{33}] + \ldots\ldots, \\
(222)\,Y_1 &= Y[4(22)(12) - 2\Delta_{12}] + \ldots\ldots, \\
(112)\,Y_1 &= Y[4(11)(12) + 2\Delta_{12}] + \ldots\ldots, \\
(122)\,Y_1 &= 2Y[(11)(22) + (12)^2 + \Delta_{22}] + \ldots\ldots, \\
(123)\,Y_1 &= 4Y(12)(13) + \ldots\ldots, \\
(223)\,Y_1 &= 2Y[(13)(22) + (12)(23)] + \ldots\ldots,
\end{aligned}
$$

where the omitted terms are of the third or lower degree.

Finally, the quartics are

$$
\begin{aligned}
&(11)J + Y\Delta_{11} + \ldots\ldots = 0,\\
&(12)J + Y\Delta_{12} + \ldots\ldots = 0,\\
&(\Delta_{11} + \Delta_{22})^2 + [(11) + (22)]^2 Y + \ldots\ldots = 0,\\
&\Delta_{23}^2 + [(11) + (22)][(11) + (33)]Y + \ldots\ldots = 0,\\
&\Delta_{13}\Delta_{23} - (12)[(11) + (22)]Y + \ldots\ldots = 0,\\
&\Delta_{23}(\Delta_{11} + \Delta_{22}) + (23)[(11) + (22)]Y + \ldots\ldots = 0,
\end{aligned}
$$

where the terms omitted are of the third or lower degree.

These quartics may obviously be expressed as the coefficients of two covariants, one of the fourth, the other of the second order. It may be proved that of the 21 quartics indicated, 15 are linearly independent. *There are thus* 15 *linearly independent relations of the fourth degree among the second derivatives of the logarithm of a hyperelliptic theta function of genus three.*

Now these relations can not be functionally independent; it is in fact clear that since the second derivatives are functions of three independent variables, the quartics, regarded as five-fold spreads in space of six dimensions, must all pass through a common three-fold. It is easy to see, by consideration of the highest-degree terms, that the surface at infinity for this three-fold is given by the vanishing of all the first minors of Δ. These six quantities all vanish if any three of them are zero, and it follows that the surface at infinity and therefore the three-fold itself are of the eighth degree.

Thus the generalization of the Kummer Quartic Surface is a certain three-fold spread of the eighth degree in space of six dimensions.

§ 4.

We next consider the non-hyperelliptic case. The function Y is not now a quadratic function of the second derivatives, and thus the third-order functions Δ_{pq}, (pq), Y, 1 are linearly independent. Hence there can be no others linearly independent of these, and therefore the quantities R_{pq} must be linearly expressible in terms of them. We write

$$R_{pq} = \sum_{h,k} c_{pq,\,hk}\,\Delta_{hk} + \sum_{h,k} d_{pq}^{(hk)}\,(hk) + d_{pq} + e_{pq}\,Y,$$

where the summation extends once to each pair of values of h, k, and substitute in the equation (2). This equation now becomes a linear function of the fourteen third-order functions, and hence it must vanish identically.

By equating its coefficients to zero there is obtained as before a set of relations among the various constants involved. It appears that there are no limitations on the constants a_{pqrs}. We may modify our theta function as before by multiplying by an exponential factor, and also we may add any linear function of the second derivatives to Y. The only additional constants that enter into the equation may be got rid of by these modifications, and therefore the only essential constants are the a's. We find that the modifications above mentioned can be made in only one way (with a trivial exception), if the equations are to be covariantive. In this case there is one fixed form, which is the general quartic

$$F \equiv \alpha_x^4 \equiv \beta_x^4 \equiv \ldots \equiv \sum_{p,q,r,s} a_{pqrs} x_p x_q x_r x_s, \qquad (p, q, r, s = 1, 2, 3).$$

The equations, in symbolic form, are given by

$$d^4\wp - 6(d^2\wp)^2 = YF + (a\alpha\beta)^2 \alpha_x^2 \beta_x^2 - \tfrac{1}{18} S, \tag{5}$$

where

$$S = (\beta\gamma\delta)(\gamma\delta\alpha)(\delta\alpha\beta)(\alpha\beta\gamma)\,\alpha_x\beta_x\gamma_x\delta_x;$$

and

$$\begin{aligned} d^2Y - 6Yd^2\wp = &- 2(ab\alpha)^2\alpha_x^2 + \tfrac{1}{6}(\alpha\beta a)^2(\alpha\beta\gamma)^2\gamma_x^2 \\ &- \tfrac{1}{72}(\alpha\beta\gamma)^2(\alpha\delta\varepsilon)^2(\delta\beta\gamma)(\varepsilon\beta\gamma)\,\delta_x\varepsilon_x. \end{aligned} \tag{6}$$

These covariant expressions are definite except in one particular. It is seen at once that Y behaves like an invariant of the third degree. Now $A \equiv \frac{1}{6}(\alpha\beta\gamma)^4$ is a similar invariant, and hence we may, if we like, take instead of Y the function $Y' = Y + \lambda A$, where λ is an absolute constant. There is then a corresponding modification to be made to the second term on the right-hand side of (6). For instance, if $Y' = Y - \frac{1}{18}A$, the identity

$$(\alpha\beta\gamma)^2(\beta\gamma a)^2\alpha_x^2 = 2(\alpha\beta a)(\alpha\gamma a)(\alpha\beta\gamma)^2\beta_x\gamma_x + 2Aa_x^2$$

shows that this second term becomes $\frac{1}{3}(\alpha\beta a)(\alpha\gamma a)\beta_x\gamma_x$.

These expressions were obtained by taking the quartic in the canonical form

$$F = x_1^4 + x_2^4 + x_3^4 + 6\Sigma h_1 x_2^2 x_3^2 + 12\Sigma p_1 x_1^2 x_2 x_3,$$

and calculating the constants directly from the equations among the coefficients of the equations of type (2). It then appeared that the part of B (we recall that B is used to denote that part of the right-hand side of (5) that does not involve Y) involving first derivatives was quadratic in the constants of F. The only available covariant was therefore $(\alpha\beta a)^2\alpha_x^2\beta_x^2$, and it was found that by adding

suitable constants to the second derivatives, and incorporating a linear function of these second derivatives into Y, this part of B could be identified with this covariant. It is interesting to note that the method followed leads to the explicit forms of the various covariants involved, and therefore is one for calculating certain covariants of the quartic. The constant terms are two such covariants for which we thus have explicit expressions. They are both given by Salmon* for the particular case in which p_1, p_2, p_3 are all zero.

We proceed to give the explicit forms of the equations:

$$\begin{aligned}
B_{1111} &= 2(h_2h_3-p_1^2)(11)+2h_2(22)+2h_3(33)-4p_1(23)+b_{1111},\\
B_{1112} &= (h_3p_3-2p_1p_2)(11)+p_3(22)-2(h_2h_3-p_1^2)(12)-2p_2(23)+b_{1112},\\
B_{1122} &= \tfrac{1}{3}(h_2+h_1h_3-4p_2^2)(11)+\tfrac{1}{3}(h_1+h_2h_3-4p_1^2)(22)+\tfrac{1}{3}(1-3h_3^2)(33)\\
&\quad +2h_3p_1(23)+2h_3p_2(31)+\tfrac{2}{3}(5p_1p_2-4h_3p_3)(12)+b_{1122},\\
B_{1123} &= -\tfrac{2}{3}(h_1p_1+p_2p_3)(11)-h_2p_1(22)-h_3p_1(33)+\tfrac{2}{3}(2h_2h_3+p_1^2-h_1)(23)\\
&\quad +\tfrac{2}{3}(h_3p_3+p_1p_2)(31)+\tfrac{2}{3}(h_2p_2+p_3p_1)(12)+b_{1123},
\end{aligned}$$

where the coefficients of $-S$ are given by

$$\begin{aligned}
18b_{1111} &= -24(h_2h_3-p_1^2)-24p_1p_2p_3+24h_2p_2^2+24h_3p_3^2,\\
18b_{1112} &= 6\{(h_3p_3-2p_1p_2)(p_1^2-h_2h_3)+2h_3h_1p_3-h_1p_1p_2-h_2p_3\},\\
18b_{1122} &= 4h_1h_2h_3^2-4h_1h_3p_1^2-4h_2h_3p_2^2-4h_3^2p_3^2+16h_3p_1p_2p_3-12p_1^2p_2^2\\
&\quad -4h_1p_2^2-4h_2p_1^2+4h_1^2h_3+4h_2^2h_3-4h_1h_2-4p_3^2,\\
18b_{1123} &= -8h_1h_2h_3p_1-6h_2h_3p_2p_3+8h_1p_1^3+8h_2p_1p_2^2+8h_3p_1p_3^2-14p_1^2p_2p_3\\
&\quad +4h_1p_2p_3-2h_1^2p_1+2p_1.
\end{aligned}$$

Also

$$\begin{aligned}
R_{11} &= -4\Delta_{11}-4h_3\Delta_{22}-4h_2\Delta_{33}-8p_1\Delta_{23}\\
&\quad +\tfrac{1}{3}[2h_1h_2h_3-2p_1p_2p_3+4h_1p_1^2+2h_2p_2^2+2h_3p_3^2+3h_1^2+h_2^2+h_3^2+1](11)\\
&\quad +\tfrac{1}{3}[4h_2(h_2h_3-p_1^2)+2(h_1h_2+h_3+p_3^2)](22)+(\bigstar)(33)\\
&\quad +\tfrac{1}{3}[8p_1(h_1-h_2h_3+p_1^2)-2p_2p_3](23)\\
&\quad +\tfrac{1}{3}[6p_1p_3h_3-2p_2(h_2h_3+2p_1^2+3h_1)](31)+(\bigstar)(12)+d_{11},\\
R_{12} &= -4p_3\Delta_{33}-8p_2\Delta_{23}-8p_1\Delta_{31}-8h_3\Delta_{12}\\
&\quad -\tfrac{1}{3}[5h_1h_3p_3+2h_1p_1p_2-h_2p_3](11)+(\bigstar)(22)\\
&\quad +\tfrac{1}{3}[7h_3(h_3p_3-2p_1p_2)+p_3](33)\\
&\quad +\tfrac{1}{3}[4p_2(3h_2h_3+p_1^2+h_1)-2h_3p_1p_3](23)+(\bigstar)(31)\\
&\quad +\tfrac{1}{3}[4h_1h_2h_3-10p_1p_2p_3+8h_1p_1^2+8h_2p_2^2+12h_3p_3^2+4h_3^2](12)+d_{12},
\end{aligned}$$

* "Higher Plane Curves," 3rd Ed., pp. 270, 273.

where

$$
\begin{aligned}
9d_{11} = {} & 4h_1h_2^2h_3^2 + 4h_1h_2h_3p_1^2 - 8h_1p_1^4 + 8p_1^3p_2p_3 + 28h_2h_3p_1p_2p_3 - 8h_2p_1^2p_2^2 \\
& - 10h_2^2h_3p_2^2 - 8h_3p_1^2p_3^2 - 10h_2h_3^2p_3^2 - 6h_1^2h_2h_3 - 20h_1p_1p_2p_3 + 6h_1^2p_1^2 \\
& + 8h_1h_2p_2^2 + 8h_1h_3p_3^2 + 6p_2^2p_3^2 - 2h_2^2p_1^2 - 2h_3^2p_1^2 + 2h_2^3h_3 + 2h_2h_3^3 \\
& + 2h_2p_3^2 + 2h_3p_2^2 - 2h_2h_3 + 2p_1^2, \\
9d_{12} = {} & - 8h_1h_2h_3^2p_3 - 2h_1h_2h_3p_1p_2 - 6h_3p_1p_2p_3^2 + 4p_1^2p_2^2p_3 + 2h_1h_3p_1^2p_3 \\
& - 4h_1p_1^3p_2 + 2h_2h_3p_2^2p_3 - 4h_2p_1p_2^3 + 2h_3^2p_3^3 + 2h_1^2h_3p_3 + 2h_2^2h_3p_3 \\
& + 2h_1^2p_1p_2 + 2h_2^2p_1p_2 + 2h_3^3p_3 - 4h_3^2p_1p_2 - 6h_1p_2^2p_3 - 6h_2p_1^2p_3 - 2p_3^3 \\
& + 4h_1h_2p_3 - 2h_3p_3 + 2p_1p_2.
\end{aligned}
$$

The expressions omitted are obtained by appropriate interchange of the suffixes 1, 2, 3.

We shall find it convenient to use the symbolic notation for most of the remainder of our work. We use $p_x^3 \equiv q_x^3 \equiv \ldots.$ for the form whose coefficients are third derivatives, and ξ_x^4 for fourth derivatives; also L_x, M_x^2, N_x^3, ... are used to denote first, second, etc., derivatives of Y.

It is easy to differentiate a symbolic expression. For example, suppose we have a covariant linear in second derivatives, say $(a\alpha\beta)^2\alpha_x^2\beta_x^2$. The first derivative of this is $(p\alpha\beta)^2p_x\alpha_x^2\beta_x^2$; the second is

$$(\xi\alpha\beta)^2\alpha_x^2\beta_x^2\xi_x^2 \equiv \xi_y^2\xi_x^2\alpha_x^2\beta_x^2.$$

We now use the polarized form of (5) and can substitute for ξ at once.

The differential coefficient of $(ab\alpha)^2\alpha_x^2$ is $2(pa\alpha)^2p_x\alpha_x^2$. Its second derivative is

$$2(pq\alpha)^2p_xq_x\alpha_x^2 + 2(\xi a\alpha)^2\xi_x^2\alpha_x^2.$$

This may be expressed in terms of second derivatives and Y only, as soon as we know the expression for $(pqu)^2p_xq_x$, that is to say, the expression for (3) in symbols.

We now proceed to get some equations that we need in symbolic form. We first take (5) in completely polarized form, $d^{(1)}d^{(2)}d^{(3)}d^{(4)}\varphi = \ldots.$, and perform on it the operation $d^{(5)}$. We then interchange $d^{(4)}$ and $d^{(5)}$ and subtract. We thus obtain an equation which is linear in second derivatives, linear in third derivatives, and linear in first derivatives of Y. This is the polarized form of

$$-6(pau)p_x^2a_x = (p\alpha\beta)^2(p\beta u)\alpha_x^2\beta_x + (Lau)\alpha_x^3, \tag{7}$$

where u has been written for $(x^{(4)}x^{(5)})$. In exactly the same way we derive from (7) the equation

$$4(pqu)^2 p_x q_x = 2(\xi a u)^2 \xi_x^2 + \tfrac{1}{3}(\xi\alpha\beta)^2(\xi\beta u)^2\alpha_x^2 \\ + \tfrac{2}{3}(\xi\alpha\beta)^2(\xi\alpha u)(\xi\beta u)\alpha_x\beta_x + (M\alpha u)^2\alpha_x^2. \qquad (8)$$

If in this we replace fourth derivatives by their values in terms of second derivatives, and substitute from (6) for second derivatives of Y, we have equation (3) given in symbolic form.

We shall need in later work the value of the second derivative of $(abu)(abv)$. This may be obtained by polarizing from $(abu)^2$, and so we give $d^2(abu)^2$.

Now

$$d^2(abu)^2 = 2(pqu)^2 p_x q_x + 2(\xi a u)^2 \xi_x^2,$$

and therefore from (8)

$$2d^2(abu)^2 = 6(\xi a u)^2\xi_x^2 + \tfrac{1}{3}(\xi\alpha\beta)^2(\xi\beta u)^2\alpha_x^2 \\ + \tfrac{2}{3}(\xi\alpha\beta)^2(\xi\alpha u)(\xi\beta u)\alpha_x\beta_x + (M\alpha u)^2\alpha_x^2.$$

§ 5.

Exactly as in previous cases we have an equation

$$d^6\wp - 30 d^2\wp\, d^4\wp + 60(d^2\wp)^3 = X + hY,$$

where X consists of two parts, one linear in second derivatives, and the other constant, whilst h is constant, both X and h being sextic covariants. As before, we eliminate fourth and sixth derivatives by means of (5) and thus obtain an equation

$$12\left[(d^3\wp)^2 - 4(d^2\wp)^3 - (d^2\wp)(B + FY)\right] + K = X + hY, \qquad (9)$$

where

$$K = d^2B - 6Bd^2\wp + F(d^2Y - 6Yd^2\wp).$$

Now

$$B = (a\alpha\beta)^2\alpha_x^2\beta_x^2 + \text{const.},$$

therefore

$$d^2B = (\xi\alpha\beta)^2\alpha_x^2\beta_x^2\xi_x^2 \\ = 2(a\alpha\beta)^2\alpha_x^2\beta_x^2 b_x^2 + 4(a\alpha\beta)(b\alpha\beta)a_x b_x\alpha_x^2\beta_x^2 + \tfrac{1}{3}(a\gamma\delta)^2(\alpha\beta\gamma)^2\alpha_x^2\beta_x^2\delta_x^2 \\ + \tfrac{2}{3}(a\gamma\delta)^2(\alpha\beta\gamma)(\alpha\beta\delta)\alpha_x^2\beta_x^2\gamma_x\delta_x + (\sigma\alpha\beta)^2\sigma_x^2\alpha_x^2\beta_x^2 + (\alpha\beta\gamma)^2\alpha_x^2\beta_x^2\gamma_x^2\, Y;$$

and

$$6Bd^2\wp = 6(a\alpha\beta)^2\alpha_x^2\beta_x^2 b_x^2 + 6\sigma_x^4 a_x^2;$$

also

$$d^2Y - 6Yd^2\wp = -2(ab\alpha)^2\alpha_x^2 + \tfrac{1}{6}(a\alpha\beta)^2(\alpha\beta\gamma)^2\gamma_x^2 + \tau_x^2,$$

where

$$\sigma_x^4 = -\tfrac{1}{18}S,\ \tau_x^2 = \text{the constant term in (6).}$$

Hence

$$\begin{aligned}K = &-6(ab\alpha)^2\alpha_x^2 F + 4(ab\alpha)(ab\beta)\alpha_x^3\beta_x^3 + [\tfrac{1}{3}(\alpha\gamma\delta)^2(\alpha\beta\gamma)^2\alpha_x^2\beta_x^2\delta_x^2 \\ &+ \tfrac{2}{3}(\alpha\gamma\delta)^2(\alpha\beta\gamma)(\alpha\beta\delta)\alpha_x^2\beta_x^2\gamma_x\delta_x + \tfrac{1}{3}Sa_x^2 + \tfrac{1}{6}(a\alpha\beta)^2(\alpha\beta\gamma)^2\gamma_x^2 F] \\ &+ [\sigma\alpha\beta)^2\sigma_x^2\alpha_x^2\beta_x^2 + \tau_x^2 F] + Y(\alpha\beta\gamma)^2\alpha_x^2\beta_x^2\gamma_x^2 \\ = &\ P + Q, \text{ say,}\end{aligned}$$

where

$$P = -6(ab\alpha)^2\alpha_x^2 F + 4(ab\alpha)(ab\beta)\alpha_x^3\beta_x^3 + HY,$$

and

$$H = (\alpha\beta\gamma)^2\alpha_x^2\beta_x^2\gamma_x^2.$$

Thus when we know the quantities X and h, the equations (8), (9) serve to express products of third derivatives as cubic functions of second derivatives, and involving Y linearly. These equations, however, enable us to get the expression for Y in terms of second derivatives. To obtain this relation we differentiate (9) twice, and substitute from (8) for the third derivatives that occur. We thus obtain a relation of the form

$$(Y^2 + 2\Delta)F^2 = \Sigma\alpha_{pq,rs}(pq)(rs) + \Sigma\beta_{pq}(pq) + \gamma + \delta Y,$$

where the coefficients on the right side are constants. It is clear that F^2 must divide through the equation, and thus we have restrictions on X and h, which enable us to determine these quantities, with the exception of the constant term in X.

When we differentiate (9) twice and substitute for fourth derivatives, we have the equation

$$12(B + YF)^2 + d^2K - 12Kd^2\varphi = d^2X + hd^2Y.$$

We can calculate d^2K in symbols. Also, X being a covariant at most linear in the coefficients of a_x^2, we see that $d^2X - 6Xd^2\varphi$ is an expression linear in the quantities Δ_{pq}, (pq). When the above expression is written in expanded form, it appears that F^2 is a factor of all terms except those of the types $(pq)(rs)$, $(pq)Y$, (pq), 1. It must therefore also divide the terms of the types given. It follows at once that there are no terms of the type $(pq)Y$, and hence h is determined. A consideration of terms of the type $(pq)(rs)$ determines the part of X involving the coefficients of a_x^2. We obtain finally the result

$$\begin{aligned}12[(d^3\varphi)^2 - 4(d^2\varphi)^3 - (d^2\varphi)(B + FY)] + K = &-\tfrac{5}{3}HY + Z \\ &- 5(\alpha\gamma\delta)^2(\alpha\beta\gamma)^2\alpha_x^2\beta_x^2\delta_x^2 + 6(\alpha\gamma\delta)^2(\alpha\beta\gamma)(\alpha\beta\delta)\alpha_x^2\beta_x^2\gamma_x\delta_x - \tfrac{1}{3}Sa_x^2 \\ &+ \tfrac{10}{3}AFa_x^2 + \tfrac{5}{2}F(a\alpha\beta)^2(\alpha\beta\gamma)^2\gamma_x^2 - 4(a\alpha\beta)(\delta\alpha\beta)(\alpha\beta\gamma)^2 a_x\gamma_x^2\delta_x^3 \quad \ldots \quad (10)\end{aligned}$$

where

$$H = (\alpha\beta\gamma)^2 \alpha_x^2 \beta_x^2 \gamma_x^2,$$

and Z is a sextic covariant of the quartic alone, of the sixth order in the coefficients.

This equation (10), in conjunction with (8), gives expressions for all squares and products of third derivatives as cubic functions of the second derivatives, and involving Y linearly. The constant terms, the coefficients of Z, are however as yet undetermined.

The resulting equation in the above work, when F is divided out, is an expression for Y as a function of the second derivatives. This equation is

$$Y^2 - \tfrac{1}{18} A Y = -2\Delta + \tfrac{1}{6}(a\alpha\beta)^2(b\alpha\beta)^2 + \tfrac{1}{72}(a\alpha\beta)^2(\alpha\gamma\delta)^2(\beta\gamma\delta)^2 - {}^{5} D, \quad (11)$$

where D is the six-rowed determinant, the invariant of the quartic of the sixth degree in the coefficients, given by Salmon, "Higher Plane Curves," 3rd Ed., p. 265, and called by him B.

This equation was in fact calculated as follows: All the terms except those linear in second derivatives and the constant were determined in the manner indicated above. The remaining terms, being invariant, were necessarily of the form

$$n_1 (a\alpha\beta)^2 (\alpha\gamma\delta)^2 (\beta\gamma\delta)^2 + n_2 A^2 + n_3 D,$$

where n_1, n_2, n_3 were merely numerical constants. n_2 was then shown to be zero by means of the particular quartic $x_1^4 + x_2^4 + x_3^4$, and the remaining two constants were found by calculating out one of the equations for Salmon's form of the quartic, $\Sigma x^4 + 6\,\Sigma h_1 x_2^2 x_3^2$.

There yet remain to be determined the relations connecting second derivatives only. Now there are 70 functions of the fifth order involving second derivatives and Y. One relation is given by (11), and since there are only 63 linearly independent fifth-order functions, it is clear that there must be six other such relations.

We proceed to obtain these relations, to show, in fact, that the coefficients of squares and products of the u's in $Y(abu)^2$ may be linearly expressed by means of the other fifth-order functions.

Equation (6) in polarized form is

$$M_x M_y - 6\, Y a_x a_y = -2(ab\alpha)^2 \alpha_x \alpha_y + \tfrac{1}{6}(a\alpha\beta)^2 (\alpha\beta\gamma)^2 \gamma_x \gamma_y + \tau_x \tau_y.$$

We differentiate this with respect to z, then interchange y and z in the equation obtained and subtract; we thus get the equation

$$-6(Lau)a_x = -4(pa\alpha)^2(pa u)\alpha_x + \tfrac{1}{6}(p\alpha\beta)^2(\alpha\beta\gamma)^2(p\gamma u)\gamma_x \cdots, \qquad (12)$$

where u is written for (yz).

We next differentiate (12) with respect to y, interchange x and y, subtract, and then write $(xy) = u$. We thus have the equation

$$6(Mau)^2 = 4(\xi a\alpha)^2(\xi\alpha u)^2 + 4(pq\alpha)^2(pu\alpha)(qu\alpha) - \tfrac{1}{6}(\xi\alpha\beta)^2(\alpha\beta\gamma)^2(u\xi\gamma)^2.$$

In this equation we substitute for $(Mau)^2$ from (6), and for $(pq\alpha)^2(pu\alpha)(qu\alpha)$ from (8), after writing in (8), α for u, and $(u\alpha)$ for x. We thus have the relation

$$\begin{aligned}36Y(abu)^2 - 12(ab\alpha)^2(cu\alpha)^2 + (a\alpha\beta)^2(\alpha\beta\gamma)^2(bu\gamma)^2 + 6(\tau a u)^2 \\ = 6(a\xi\alpha)^2(u\xi\alpha)^2 + (\xi\alpha\beta)^2(\xi\beta\gamma)^2(\alpha\gamma u)^2 - \tfrac{1}{2}(\xi\alpha\beta)^2(\alpha\beta\gamma)^2(\xi\gamma u)^2 \\ + (M\alpha\beta)^2(\alpha\beta u)^2.\end{aligned}$$

This relation, after substitution for ξ and M, becomes

$$\begin{aligned}36Y[(abu)^2 - \tfrac{1}{3}(a\alpha\beta)^2(\alpha\beta u)^2 - \tfrac{1}{72}(\alpha\beta\gamma)^2(\alpha\beta\delta)^2(\gamma\delta u)^2] \\ = 36(ab\alpha)^2(c\alpha u)^2 + [12(a\alpha\beta)^2(b\beta\gamma)^2(\gamma a u)^2 - 6(\alpha\beta\gamma)^2(b\alpha\beta)^2(a\gamma u)^2 \\ - 3(ab\gamma)^2(\alpha\beta\gamma)^2(\alpha\beta u)^2 + 2A(abu)^2] + [-6(a\tau u)^2 + 6(a\sigma\alpha)^2(u\sigma\alpha)^2 \\ + (a\delta\varepsilon)^2(\delta\alpha\beta)^2(\varepsilon\beta\gamma)^2(\alpha\gamma u)^2 - \tfrac{1}{2}(a\delta\varepsilon)^2(\delta\alpha\beta)^2(\alpha\beta\gamma)^2(\varepsilon\gamma u)^2 \\ + \tfrac{1}{3}A(a\alpha\beta)^2(\alpha\beta u)^2 - \tfrac{1}{3}(a\delta\varepsilon)^2(a\delta\varepsilon)^2(\alpha\beta\gamma)^2(\beta\gamma u)^2] \\ + [(\sigma\alpha\beta)^2(\sigma\beta\gamma)^2(\alpha\gamma u)^2 - \tfrac{1}{2}(\sigma\alpha\beta)^2(\alpha\beta\gamma)^2(\sigma\gamma u)^2 + (\tau\alpha\beta)^2(\alpha\beta u)^2] \ldots (13)\end{aligned}$$

Again, since Y is of the second order, we may write it $\dfrac{\Phi}{\theta^2}$, where ϕ is an integral function, and $\wp$ is $-\log\theta$. Hence

$$dY = -\frac{2\Phi\theta'}{\theta^3} + \frac{\Phi'}{\theta^2}, \quad d^2Y - 6Y(d^2\wp) = -4\frac{\Phi'\theta' - \phi\theta''}{\theta^3} + \cdots.$$

Also

$$dYd^3\wp - 4Y(d^2\wp)^2 = -2\frac{\Phi'\theta' - \phi\theta''}{\theta^3}\frac{\theta'^2}{\theta^2} + \cdots.$$

and therefore

$$dYd^3\wp - 4Y(d^2\wp)^2 - (d^2Y - 6Yd^2\wp),$$

though apparently of the sixth, is really of the fourth order. Hence

$$dYd^3\wp - 4Y(d^2\wp)^2 + (ab\alpha)^2\alpha_x^2(d^2\wp)$$

is of the fourth order. The expression of this in terms of the fourth-order functions may now be computed, and we have the result

$$\begin{aligned} dYd^3\wp - 4Y(d^2\wp)^2 + (ab\alpha)^2\alpha_x^2(d^2\wp) &= \tfrac{1}{3}(a\alpha\beta)^2(b\beta\gamma)^2 \\ &+ \tfrac{2}{9}(a\alpha\beta)^2(b\alpha\beta)(\alpha\beta\gamma)\,b_x\gamma_x^3 - \tfrac{4}{3}\Delta F - \tfrac{1}{18}(a\alpha\beta)^2(b\alpha\beta)^2F \\ &+ \tfrac{2}{3}Y(a\alpha\beta)^2\alpha_x^2\beta_x^2 + \text{an expression of the third order.} \qquad (14) \end{aligned}$$

Also, if in (7) we replace u by q, and multiply by q_x^2, we have

$$(p\alpha\beta)^2(p\beta q)\,\alpha_x^2\beta_x p_x^2 + (L\alpha p)\,\alpha_x^3 p_x^2 = 0,$$

or

$$(pq\beta)^2(p\alpha\beta)\,\alpha_x^3\beta_x q_x + (Lp\alpha)\,p_x^2\alpha_x^3 = 0.$$

From this last we readily deduce

$$(Lpu)\,p_x^2 = (pq\alpha)^2(p\alpha u)\,q_x\alpha_x. \qquad (15)$$

By means of (8), the right-hand side of (15) may be expressed in terms of functions of the second order, and then (14) and (15) enable us to express any product of a first derivative of Y and a third derivative of $\wp$ as a cubic function of the functions of the second order.

We now obtain two equations from (8) in exactly the same way that (7) and (8) were determined from (5). These two equations are

$$6(\xi qu)^3\xi_x = (\eta\alpha\beta)^2(\eta\beta u)^2(\eta\alpha u)\,\alpha_x + (N\alpha u)^3\alpha_x, \qquad (16)$$

$$-6(\xi\xi' u)^4 = (\theta\alpha\beta)^2(\theta\alpha u)^2(\theta\beta u)^2 + (P\alpha u)^4, \qquad (17)$$

where

$$\eta_x^5 = d^5\wp, \quad \theta_x^6 = d^6\wp, \quad P_x^4 = d^4Y.$$

When (17) is expanded it becomes

$$\begin{aligned} (abu)^2(cdu)^2 + \tfrac{3}{2}Y(a\alpha u)^2(b\alpha u)^2 + \tfrac{3}{2}(c\alpha\beta)^2(a\alpha u)^2(b\beta u)^2 + \tfrac{5}{36}\Delta(\alpha\beta u)^4 \\ - \tfrac{1}{3}(abu)^2(c\alpha\beta)^2(\alpha\beta u)^2 - \tfrac{2}{3}(ab\alpha)^2(\alpha\beta u)^2(c\beta u)^2 = \ldots\ldots \qquad (18) \end{aligned}$$

where the terms on the right are at most quadratic in the second order functions.

Now we showed originally that there were 22 relations among the functions of the sixth order that were rational functions of the second derivatives and at most linear in Y. The equation (18) gives 15 of these relations, by equating to zero the various coefficients of powers and products of the u's. There are six others given similarly by equation (13). Also if in (13) we replace u by c, we have a relation independent of those already enumerated, which expresses $Y\Delta$ in terms of second derivatives. We have thus altogether 22. It is almost demonstrable that there are no others, and in fact that all other relations among second and third derivatives, involving Y and its first derivatives, may be derived

by algebraic processes from those given. In particular, there are relations among the second derivatives only. There are apparently none of order so low as 4. They may be obtained by eliminating Y from the 22 relations mentioned. For example, we obtain 15 quintic relations among second derivatives only, by elimination of Y from the equations derived from (13). Similarly we may obtain from (18) and (13) together 201 sextics, though these are not necessarily linearly independent. We notice, however, that the highest-degree terms of all these relations are homogeneous quadratics in the quantities Δ_{pq}, with coefficients quadratic functions of second derivatives. It follows that if we take second derivatives as coordinates in space of six dimensions, all the five-folds mentioned pass doubly through the surface of the eighth order at infinity given by the vanishing of all the first minors of Δ. Now we know *a priori* that these relations must have a common three-fold, and it is clear therefore that this three-fold must be either of the eighth or of the sixteenth order; it seems highly probable that it is of the sixteenth order, and the space at infinity is a trope. This three-fold is the generalization of the Kummer Quartic, which arises when $p = 2$, or of the non-singular cubic curve, for $p = 1$. We propose to consider it more in detail later.

BRYN MAWR COLLEGE.

Differential Equations Admitting a Given Group.

By J. Edmund Wright.

Let there be given a continuous r-parameter group in the n variables $x_1, x_2, \ldots, x_{n-1}, y$, and let the x's be functions of y. Suppose a system of $n-1$ differential equations is given by $I_1 = 0$, $I_2 = 0$, $\ldots$, $I_{n-1} = 0$, where the I's are differential invariants of the group, of orders $a_1, a_2, \ldots, a_{n-1}$ respectively. Then their complete solution involves N arbitrary constants. But since the system is invariant under the given group, it follows that from a solution involving $N-r$ constants we can deduce the general solution, provided the particular solution is not invariant under any subgroup of the given group.

The general solution consists of a set of curves, ∞^N in number, in space of n dimensions, and these curves are merely interchanged under the operations of the group, so that, *e. g.*, from a single curve we may derive by means of the transformations of the group a set of ∞^r curves, provided that the single curve is not invariant under any transformation of the group. All of these are solutions of the differential equations.

Now any curve may be expressed by equations of the type $x_i = f_i(t)$, where f is an appropriate function of t, and t is parametric. If the group be extended by adding the variable t, and assuming $x_1, x_2, \ldots, x_{n-1}, y$ to be functions of t, whilst t is an invariant, we get $n-1$ equations in n dependent variables, all of which are invariants of the group as thus modified.

In cases that frequently arise, *e.g.*, in differential geometry or in dynamics, the independent variable is an invariant of the group, and may itself be taken as t. In other cases we change the independent variable as indicated, and we then need an equation defining t. This equation must be one among the n dependent variables and t, and if it is quite general it may involve derivatives of the dependent variables. Suppose it to be solved for t; then since t is an invariant of the group, if the new system is to remain an invariant system, we

must have a relation of the form $I_n = t$, where I_n is an invariant of the group. For the rest we may choose I_n arbitrarily except that it must be independent of $I_1, I_2, \ldots, I_{n-1}$. We have finally n equations for the n dependent variables as functions of t. Their solution depends on N' constants, where N' may be greater than N, but this gives not only the integral curves of the original system, but also the parametric representation of these integral curves in terms of a particular parameter t. This accounts for the additional constants involved.

Now if I be any invariant of the group, $\frac{dI}{dt}$ is also an invariant. If we regard $\frac{dI}{dt}$ as depending on I, then it may be proved that the group contains precisely n independent invariants in addition to t. We indicate the proof if $r < n$. Suppose that $r - h$ of the operators of zero order are unconnected, $r - h_1$ of the operators of unit order unconnected and so on. Then $h \geq h_1 \geq h_2 \ldots$ and, finally, for some value of λ, h_λ is zero. We have $n - r + h$ invariants of zero order, $2n - r + h_1$ of order ≤ 1, and so on. We thus have $pn - r + h_p$ invariants of order $\leq p$ and $(p-1)n - r + h_{p-1}$ invariants of order $\leq p - 1$. Thus there are precisely $n - (h_{p-1} - h_p)$ invariants of order p. Now there are $n - r + h$ invariants of zero order, and therefore this number of invariants of unit order are of the form $\frac{dI}{dt}$. Thus the number of new first-order invariants is

$$n - h + h_1 - (n - r + h) = r + h_1 - 2h.$$

The number of new p-th order invariants is

$$n - (h_{p-1} - h_p) - [n - (h_{p-2} - h_{p-1})] = h_{p-2} + h_p - 2h_{p-1}.$$

Thus, finally, the total number of independent invariants is

$$(n - r + h) + (r + h_1 - 2h) + (h + h_2 - 2h_1) + \ldots = n.$$

Let these independent invariants be $X_1, X_2, \ldots, X_n$; then the solution of the differential equations $X_i = F_i(t)$ in the original variables will contain r arbitrary constants. Further, any system of equations of the type $I = 0$ may be expressed in terms of $t, X_1, \ldots, X_n, \ldots, \frac{d^i X_k}{dt^i}, \ldots$. Thus the original system may be integrated in two steps. First we have a set of n equations in the n variables X, and t. If these are integrated they give solutions

$$X_1 = F_1(t),\ X_2 = F_2(t),\ \ldots, X_n = F_n(t),$$

which involve N' constants. Now, regarding the X's as functions of the x's, y and their derivatives, we have to integrate a system of invariant equations of order r. But the solutions of this last system are the different curves derived from a single curve by the operations of the group. We therefore need to know a single solution to determine the complete set, provided that the single solution is not invariant under any operation of the group.

For example, consider the group of rotations about the origin in space of three dimensions; the differential operators are

$$y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y},\ z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z},\ x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x},$$

and if t is the independent variable the invariants of the group are

$$\Sigma x^2,\ \Sigma x'^2,\ \Sigma x''^2,\ \ldots,\ \Sigma xx',\ \Sigma xx'',\ \ldots,$$

$$\Delta \equiv \begin{vmatrix} x & y & z \\ x' & y' & z' \\ x'' & y'' & z'' \end{vmatrix}, \text{ etc.}$$

We may choose as our invariants X_1, X_2, X_3,

$$X_1 = \Sigma x^2,\ X_2 = \Sigma x'^2,\ X_3 = \Delta,$$

and *then any system of equations invariant under the group will involve for their final solution the determination of a single solution of*

$$X_1 = f_1(t),\ \ X_2 = f_2(t),\ \ X_3 = f_3(t).$$

We now take $x\sqrt{f_1(t)}$ instead of x, $y\sqrt{f_1(t)}$ instead of y, and $z\sqrt{f_1(t)}$ instead of z, and introduce instead of t a new variable t_1 defined by

$$dt_1 = \sqrt{\frac{f_2(t)}{f_1(t) - \frac{1}{4}f_1'(t)/\sqrt{f_1(t)}}}\,dt.$$

With these changes our system of equations becomes

$$\Sigma x^2 = 1,\ \ \Sigma x'^2 = 1,\ \ \Delta = f(t),$$

where we now drop the suffix from t_1; and we shall assume that f is not zero. We see that $\Sigma xx' = 0$, hence x, y, z, and x', y', z', are the direction cosines of two straight lines at right angles. Also

$$\Sigma xx'' = -\Sigma x'^2 = -1.$$

Thus

$$\Sigma x(x'' + x) = 0,\ \ \Sigma x'(x'' + x) = 0,$$

and therefore the line perpendicular to the two given lines must have its

direction cosines proportional to $x'' + x$, etc. It follows from the third equation that these direction cosines are precisely $(x'' + x)/f(t)$, etc. From the known relations that exist among the direction cosines of three mutually perpendicular straight lines we have at once

$$\left(\frac{x'' + x}{f}\right)^2 + x'^2 + x^2 = 1, \tag{1}$$

a differential equation of the second order for x. If this be differentiated, it gives the linear equation of the third order

$$x''' - \frac{f'}{f}x'' + (1 + f^2)x' - \frac{f'}{f}x = 0; \tag{2}$$

(it may easily be seen that the apparent solution $x'' + x = 0$ must be excluded). We determine any particular solution of (2) that satisfies (1), and then by elimination we get a first-order equation for y. If we can find any particular solution of the equation for y, z is determinate, and then the complete solution of the set of equations may be at once written down.

As an illustration, suppose that $x = kt$ is a particular solution of (1); then $f = kt/(1 - k^2 - k^2t^2)^{\frac{1}{2}}$. Corresponding particular values of y and z are given by

$$f(1 - k^2)^{\frac{1}{2}}y = kt\,(\cos u + kf \sin u),$$
$$f(1 - k^2)^{\frac{1}{2}}z = kt\,(\sin u - kf \cos u),$$

where $\sin ku = kt/(1 - k^2)^{\frac{1}{2}}$.

If we call these particular values x_0, y_0, z_0, the general solution of the equations is

$$x = l\,x_0 + m\,y_0 + n\,z_0,$$
$$y = l'\,x_0 + m'\,y_0 + n'\,z_0,$$
$$z = l''x_0 + m''y_0 + n''z_0,$$

where the constants l, m, n, etc. are the direction cosines of three mutually perpendicular straight lines.

BRYN MAWR COLLEGE, *November*, 1908.

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THE SPECTRA OF SULPHUR DIOXIDE

FRANCES LOWATER

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THE SPECTRA OF SULPHUR DIOXIDE

By FRANCES LOWATER

I. THE ABSORPTION SPECTRUM

I. HISTORY

The absorption spectrum of sulphur dioxide was investigated in the ultraviolet region by W. A. Miller[1] in 1863. He inclosed the gas in a previously exhausted brass tube, two feet in length, the ends being closed by quartz plates. His source of radiation was the silver spark. He found that the silver spectrum was transmitted from scale-reading 96.5 to 110.5, at which point it was abruptly cut off. Estimated from a curve plotted from his maps, this range appears to be from λ 420 to 345 $\mu\mu$. He does not say at what pressure the gas was inclosed in the tube.

In 1883 Professors Liveing and Dewar[2] found that sulphur dioxide produced an absorption band "very marked between R (3179) and wave-length 2630, and a fainter absorption extending on the less refrangible side to O(3440), and on the other side to the end of the range photographed, wave-length 2300." They used as source of light the iron spark, and obtained the spectrum by means of a spectrometer having a single quartz prism and quartz lenses.

The present investigation was undertaken at the suggestion of Professor J. S. Ames, of Johns Hopkins University, and carried out at Bryn Mawr College.

II. APPARATUS AND METHOD

For the greater part of the work the gas was inclosed in a steel tube, 207 cm long, having its ends closed with quartz plates. It was provided with two pin valves for exhaustion of the tube and admission of the gas. The spectral apparatus was a quartz spectrograph of middle size by Fuess, used with a Rowland plane reflection grating having 14,438 lines to the inch.

[1] *Phil. Trans.*, **152**, II, 861–887, 1863.

[2] *Proc.. R. S*, **35**, 71–74, 1883.

In the region λ 690 to 390 $\mu\mu$ the carbon arc was used as source of light. In the region λ 410 to 210 $\mu\mu$ the source of light was the spark of an alloy of cadmium and zinc in proportions of their atomic weights. The beam was made parallel by a quartz lens before it entered the tube; on emergence it was brought, by another quartz lens, to a focus on the slit of the spectrograph. To obtain a continuous background with this spark, since no alternating current was available, the current from ten secondary cells was supplied to the primary of a ten-inch induction coil, and a capacity of 0.03 mfd. was placed across the terminals of the secondary, in parallel with the spark.

Before use the steel tube was thoroughly cleaned with hot potassium hydroxide and distilled water, and then thoroughly dried. The tube was exhausted by a water aspirator to about $1\frac{1}{2}$ cm pressure, filled with sulphur dioxide to a pressure greater than one atmosphere, and again exhausted; after this process had been repeated several times, the tube was filled with sulphur dioxide to the desired pressure. The sulphur dioxide was obtained from liquid sulphur dioxide; the high temperature of liquefaction (—10° C.) of this gas insures its purity from other gases except air, which may be present in the gas above the liquid. Before using any of the gas a considerable quantity was allowed to escape to insure that the following supply of gas should be free from air. Before admission into the tube, the sulphur dioxide was passed through a tube containing phosphorus pentoxide to insure its dryness.

The photographic plates used were Seed's No. 27 Gilt Edge. An exposure of two hours was given for the absorption spectrum in every case. A comparison spectrum was photographed on the same plate as the absorption spectrum immediately above or below the latter.

Standard wave-lengths were obtained from the lines of *Cd*, *Zn*, *Pb*, and *Fe*, which were transmitted in sufficient numbers in the absorption spectrum, the *Pb* and *Fe* being present as impurities; by this means errors were avoided that might arise from disturbance of the apparatus in changing from the arrangement for the absorption spectrum to that for the comparison or another standard spectrum.

The absorption spectrum of sulphur dioxide in the violet and ultra-violet regions was found for pressures of three atmospheres, two atmospheres, and one atmosphere, $1\frac{1}{2}$ cm, 0.45 cm, and 0.13 cm. The wave-lengths were determined by measurements made on the photographic plates in the usual way. The dividing engine used for this purpose was one by Gaertner, on which readings could be made to 0.0001 mm; that is, to a greater accuracy than settings could be made on the bands. The reduction factor was roughly 32 tenth-meters to 1 mm.

III. RESULTS

In the region λ 690 to 390 $\mu\mu$ no absorption bands are found. In the region λ 410 to 210 $\mu\mu$ the photographs show that the absorption spectrum, except at very low pressures of the gas, consists of one very wide band and a number of comparatively narrow bands of different widths and intensities. Tables I and II give the wave-lengths and intensities of the bands. The intensities are estimated by eye from the photographic plates; the scale is from 10 to $\frac{1}{2}$, 10 being the maximum and applied to bands at whose center of gravity none of the continuous background is transmitted. In many cases it is difficult to obtain accurate values of the wave-lengths; in some cases this is due to the width of the band—e. g., 3, 8, or 11 tenth-meters, while in other cases it is due to the presence of a metal line which falls within the absorption band and is strong enough to be transmitted when the continuous background is absorbed. This limitation in accuracy is apparent on comparing the readings for the same line as given in parallel columns in the tables which follow.

A tube filled with oxygen at one atmosphere's pressure and sulphur dioxide at one atmosphere's pressure gave the same spectrum as the tube filled with sulphur dioxide only at one atmosphere's pressure.

In the tables, s. denotes sharp, b. broad, n. narrow, h. hazy, i.d. ill defined, and v. very.

TABLE I

Absorption Spectrum of Sulphur Dioxide at Different Atmospheric Pressures

λ for 3 Atmos. Pressure	Intensity and Character	λ for 2 Atmos. Pressure	Intensity and Character	λ for 1 Atmos. Pressure	Intensity and Character
3881.7	9b.	3881.5	6	3881.8	1
3878.4	2s.	3878.7	3	3878.8	1n.
3828.3	10b.	3828.5	6	3828.5	2
		3825.3	3	3825.2	1n.
3776.3	3	3776.3	2	3776.4	½
3750.6	10b.	3751.1	8	3751.0	3b.
		3747.0	5s.	3747.0	2
3701.9	10v.b.	3701.7	10b.	3701.3	5s.
				3698.9	2n.
3657.4	2s.	3657.5	3n.	3657.6	1
3654.4	1	3654.4	3	3654.1	1
3650.6	1s.	3650.6	3n.	3650.7	1
3635.4	4b.	3635.4	4b.	3636.2	1b.
		3628.4	1		
		3623.5	2		
3593.9	4b.	3594.2	4v.b.	3594.2	1
3579.0	8b.	3579.1	5b.	3579.2	2
3532.8	9b.	3532.4	5b.	3532.5	1
		3529.6	3n.	3529.6	½
		3522.2	1		
		3512.3	½		
3509	f.b.	3510.1	1		
		3507.2	1		
3504	f.n.	3503.5	½		
3494.2	1	3494.1	1		
3490.1	1	3490.2	1		
3486.8	1	3486.2	½		
3474.7	1s.	3474.8	½		
3442.6	4b.	3443.2	4b.	3443.2	1
3434.1	1	3435.1	2		
3431.7	1	3432.2	2		
3423.9	2				
3422.6	3	3422.1	1		
3421.2	5	3421.3	2		
3418.7	5	3418.7	2		
3416.8	3	3417.1	2		
3414.3	2				
3412.4	2				
		3406.9	½		
3401.2	2				
3399.8	2				
3398.4	5	3398.3	1i.d.		
		3395.9	5i.d.		
3394.8	8				
		3393.9	4i.d.		
3392.5	6	3392.0	4i.d.		
		3389.3	4		
3386.7	7	3387.0	5		
3380	beginning of wide band	3384.7	4		
		3378.0	8		

TABLE I—*Continued*

λ for 3 Atmos. Pressure	Intensity and Character	λ for 2 Atmos. Pressure	Intensity and Character	λ for 1 Atmos. Pressure	Intensity and Character
		3375.9	9		
		3372.0	9	3372.1	2s.
		3365.9	9	3364.1	4s.
		3358.4	9	3358.7	8s.
		3351	beginning of wide band	3350.8	4s.
				3338.4	7
				3333.4	10
				3330	beginning of wide band

TABLE II

Absorption Spectrum of Sulphur Dioxide at Low Pressures.

Length of Column of Gas=207 cm				Length of Column of Gas=20 cm	
λ for 1½cm. pressure	Intensity and Character	λ for 0.13 cm. Pressure	Intensity and Character	λ for 1.35 cm. Pressure	Intensity and Character
3226.2	2				
3211.4	1				
3207.6	1				
3203.4	1				
3198.3	3				
3195.4	4				
3190.3	1				
3187.1	1			3178.6	f.
3180.6	9	3180.7	2		
3171.5	8 n.h.				
3166.2	8 n.h.				
3157.5	10 h.				
3152.7	10 h.	3151.9	2		
		3149.8	½s.	3150.0	½
		3147.8	½s.		
3146.6	10 h.	3145.3	½		
		3143.1	½s.		
		3137.7	½		
3134	beginning of wide band	3131.0	5n.		
		3128.7	2s.	3129.3	1
		3125.7	2s.	3124.6	1
		3120.3	4n.		
		3111.3	5s.		
		3105.8	7	3104.7	6
		3101.4	4s.		
				3093.2	1s.
				3089.7	1s.
		3086.0	10	3086.2	8s.
				3084.2	2s.
				3082.4	2s.

TABLE II.—*Continued*

Length of Column of Gas = 207 cm				Length of Column of Gas = 20 cm	
λ for $1\frac{1}{2}$ cm Pressure	Intensity and Character	λ for 0.13 cm Pressure	Intensity and Character	λ for 1.35 cm Pressure	Intensity and Character
		3064.9	10	3063.4	9
		3043.9	10	3042.6	9
		3022.6	10	3021.7	10
		3003.6	10 i.d.	3001.4	10
		2988	10 i.d.		
				2981.5	10
		2978	10 i.d.		
		2968	beginning of wide band	2962.0	10
				2943.0	10
				2927.	beginning of wide band
		2715.3	end of wide band		
		2701.5	9	2700.3	end of wide band
		2693.5	7	2692.3	9 i.d.
		2684.8	9	2683.7	10
		2676.7	7	2676.3	10
		2669.5	9	2668.6	10
		2660.1	8	2659.4	10
		2654.9	7	2653.2	9
		2653.3	7		
		2647 1	8	2646.6	9
		2643.0	7	2642.9	8
		2638.0	8	2637.3	9
		2633.0	7n.	2632.7	8
		2627.5	5	2627.1	7
		2623.1	5		
		2620.9	5	2621.2	8
		2616.9	5		
		2615.4	5		
		2613.7	7s.	2613.8	7
		2611.4	4		
		2596.8	5	2596.2	5
		2591.4	4	2590.8	3
		2585.2	3		
				2583.5	4
		2582.7	3		
		2512.2	3		
		2495.9	4	2496.2	2
		2478.1	3	2477.6	1
		2471.6	3	2471.4	1
2467	end of wide band				
		2464.3	2		
2456.	9 i.d.	2454.1	7	2454.5	2
		2448.3	6	2447.9	1

TABLE II.—*Continued*

Length of Column of Gas=207 cm				Length of Column of Gas=20 cm	
λ for $1\frac{1}{2}$ cm Pressure	Intensity and Character	λ for 0.13 cm Pressure	Intensity and Character	λ for 1.35 cm Pressure	Intensity and Character
2439	9 v.i.d.				
		2433.1	3		
2415.5	9 i.d.				
2401	9 i.d.	2401.0	2		
2397	8 i.d.	2397.5	1		
2379	8 i.d.				
2372	8 i.d.				
2367	8 i.d.				
2349	8 i.d.				
2345.5	8 i.d.				
2339	6 i.d.				
2327.5	7 i.d.				
2324	7 i.d.				
		2318.4	5		
		2308.7	6n.		
2304	6 i.d.	2303.2	8		
2297	7 i.d.	2298.0	9	2298.4	6
		2290.5	4		
2290	beginning of second wide band				
		2277.5	10	2278.4	9n.
		2269.2	6		
				2258.6	10n.
		2250.4	beginning of second wide band	2251.0	beginning of second wide band

IV. DISCUSSION

The following changes in the absorption spectrum with reduction of pressure may be noticed; they are evident from the plates.

1. As the pressure is reduced from three atmospheres to two, and from two to one, the bands become narrower and fainter, and the less refracted end of the very wide continuous band retreats toward the shorter wave-lengths, this part of the continuous absorption being replaced by narrow bands.

2. At the low pressures—namely $1\frac{1}{2}$, 0.45, 0.13 cm—the above changes are more marked; the narrow bands existing at one or more atmosphere's pressure have entirely disappeared; the wide continuous band has retreated not only from the longer wave-lengths but also from the shorter; and there is very little absorption beween

λ 257 and 230 $\mu\mu$. At the lowest pressure used, a pressure somewhat less than 0.13 cm, the wide continuous band is entirely broken up into narrow bands.

The shortest wave-length photographed was 210 $\mu\mu$; from this wave-length to 230 $\mu\mu$ the absorption decreases with the pressure, but is ill-defined, probably on account of the weakness of the continuous background.

A set of photographs has been taken of the absorption by a column of gas 20 cm in length with the gas at a pressure of 1.35 cm. Since the product

$$(\text{pressure of gas})\times(\text{length of column of gas})=207\times0.13=20\times1.35\ ,$$

the number of molecules which the beam meets in traversing a column of gas 207 cm long at a pressure of 0.13 cm is the same as it meets in traversing a column 20 cm long at a pressure of 1.35 cm. Since the numbers of molecules met in the two cases is the same we might expect the absorption to be the same, provided the physical condition of the molecules is the same. It was found that the absorption spectrum obtained from the column of gas 20 cm long at a pressure of 1.35 cm, corresponds very closely with that obtained from the column of gas 207 cm long at a pressure of 0.03 cm, but certain bands in the former are shifted toward the more refracted end of the spectrum; this is obvious from the photograph.

Photographs with this short column at pressures of 1.0 and 0.53 cm show the wide continuous band being gradually broken up into narrow bands. It is intended to extend this part of the work at the earliest opportunity.

Any mathematical relation between the wave-lengths of the bands or their reciprocals is obscure, particularly at the pressures of one or more atmospheres. The reciprocals of the wave-lengths with their differences are shown in Table III.

The differences of the frequencies suggest that the bands are arranged in groups with roughly equal differences between the first bands of successive groups; or we may regard the bands as arranged in series with roughly equal differences between the reciprocals of successive members of a series. Where the members of a series are scattered in Table III, they have been collected at the foot of

TABLE III
Wave-Lengths and Their Reciprocals
at pressure of 2 atmospheres

λ	$\frac{1}{\lambda} \times 10^7$	Successive Diff'r'nces	Group Differ.	λ	$\frac{1}{\lambda} \times 10^7$	Successive Diff'r'nces	Group Diff'r'nces
3881.5	2576.3			3486.2	2868.5		
		1.9				9.4	
3878.7	2578.2		35.7	3474.7	2877.9		
		33.8				26.4	26.4
3828.5	2612.0			3443.2	2904.3		
		36.0	36.0			6.8	
3776.4	2648.0			3435.1	2911.1		
		17.8				2.5	
3751.2	2665.8		20.7	3432.2	2913.6		
		2.9				8.6	
3747.2	2668.7			3422.1	2922.2		22.2
		32.9	32.9			0.7	
3701.5	2701.6			3421.3	2922.9		
		32.5	32.5			2.2	
3657.5	2734.1			3418.7	2925.1		
		2.3				1.4	
3654.4	2736.4			3417.1	2926.5		
		2.9				8.7	
3650.6	2739.3			3406.9	2935.2		
		11.4	25.7			8.4	
3635.4	2750.7			3398.3	2943.6		
		5.3				1.1	
3628.4	2756.0			3395.9	2944.7		
		3.8				1.8	26.0
3623.5	2759.8			3393.9	2946.5		
		22.6				1.7	
3594.0	2782.4		34.2	3392.0	2948.2		
		11.6				2.3	
3579.1	2794.0			3389.3	2950.5		
		36.9	36.9			2.0	
3532.4	2830.9			3387.0	2952.5		
		2.4				2.0	
3529.5	2833.3			3384.7	2954.5		
		5.8				5.8	
3522.2	2839.1			3378.0	2960.3		
		8.0				1.9	
3512.3	2847.1		23.4	3375.9	2962.2		25.1
		1.8				3.4	
3510.1	2848.9			3372.0	2965.6		
		2.4				5.4	
3507.2	2851.3			3365.9	2971.0		
		3.0				6.6	
3503.5	2854.3			3358.4	2977.6		
		7.7					
3494.0	2862.0						
		3.2	23.6				
3490.2	2865.2						
		3.3 (9.4)					

SERIES OF BANDS

$\frac{1}{\lambda}\times10^7$	Difference	$\frac{1}{\lambda}\times10^7$	Difference	$\frac{1}{\lambda}\times10^7$	Difference
I		II		III	
				2830.9	
		2847.1			23.4
2851.3			21.4	2854.3	
	26.6	2868.5			
2877.9			45.1		68.6
	26.4		=22.5×2		=22.9×3
2904.3					
	22.2	2913.6			
2926.5			21.6	2922.9	
	26.0	2935.2			23.6
2952.5			19.3	2946.5	
	25.1	2954.5			24.5
2977.6				2971.0	

AT PRESSURE OF 1½ CM

λ	$\frac{1}{\lambda}\times10^7$	Successive Diff'r'nces	Group Diff'r'nces	λ	$\frac{1}{\lambda}\times10^7$	Successive Diff'r'nces	Group Diff'r'nces
3226.2	3099.6			2456	4071.5		
		14.3				28.5	28.5
3211.4	3113.9		18.0	2439	4100		
		3.7				40	40
3207.6	3117.6			2415.5	4140		
		4.1				25	
3203.4	3121.7			2401	4165		32
		5.0				7	
3198.3	3126.7			2397	4172		
		2.8	20.0			31.5	31.5
3195.4	3129.5			2379	4203.5		
		5.0				12.5	
3190.3	3134.5			2372	4216		23.5
		3.1				11	
3187.1	3137.6			2367	4225		
		6.5				32	32
3180.6	3144.1			2349	4257		
		9.0	20.8			6.5	
3171.5	3153.1			2345.5	4263.5		18
		5.3				11.5	
3166.2	3158.4			2339	4275		
		8.7				21.5	
3157.5	3167.1			2327.5	4296.5		28
		4.8	19.6			6.5	
3152.7	3171.9			2324	4303		
		6.1				37	37
3146.6	3178.0			2304	4340		
						14	
				2297	4354		

SERIES OF BANDS

$\frac{1}{\lambda}\times 10^7$	Difference	$\frac{1}{\lambda}\times 10^7$	Difference	$\frac{1}{\lambda}\times 10^7$	Difference
I		II		III	
3099.6		3113.9		3117.6	
	22.1		20.6		20.0
3121.7		3134.5		3137.6	
	22.4		18.6		20.8
3144.1		3153.1		3158.4	
	23.0		18.8		19.6
3167.1		3171.9		3178.0	

AT PRESSURE OF 0.13 CM

λ	$\frac{1}{\lambda}\times 10^7$	Successive Differences	Group Differences
3180.7	3144.0		
		28.7	
3151.9	3172.7		
		2.1	
3149.8	3174.8		
		2.0	
3147.8	3176.8		
		2.5	
3145.3	3179.3		
		2.4	
3143.1	3181.7		
		5.3	
3137.7	3187.0		20.0
		6.9	
3131.0	3193.9		
		2.3	
3128.7	3196.2		
		3.1	
3125.7	3199.3		
		5.5	
3120.3	3204.8		20.5
		9.3	
3111.3	3214.1		
		5.7	
3105.8	3219.8		
		4.6	
3101.4	3224.4		20.6
		16.0	
3086.0	3240.4		
		22.3	22.3
3064.9	3262.7		
		22.6	22.6
3043.9	3285.3		
		23.1	23.1
3022.6	3308.4		
		20.9	20.9

λ	$\frac{1}{\lambda}\times 10^7$	Successive Differences	Group Differences
3003.6	3329.6		
		17.3	17.3
2988	3343.6		
2701.5	3701.6		
		11.0	
2693.5	3712.6		
		12.1	
2684.8	3724.7		23.3
		11.2	
2676.7	3735.9		
		11.1	
2669.5	3746.0		21.3
		13.2	
2660.1	3759.2		
		7.4	
2654.9	3766.6		
		2.3	
2653.3	3768.9		24.4
		8.8	
2647.1	3777.7		
		5.9	
2643.0	3783.6		
		7.2	
2638.0	3790.8		
		7.1	22.3
2633.0	3797.9		
		8.0	
2627.5	3805.9		

AT PRESSURE OF 0.13 CM (CONTINUED)

λ	$\frac{1}{\lambda}\times 10^7$	Successive Differences	Group Differences	λ	$\frac{1}{\lambda}\times 10^7$	Successive Differences	Group Differences
		6.4		2464.3	4057.9		
2623.1	3812.3					16.9	
		3.2		2454.1	4074.8		26.6
2620.9	3815.5					9.7	
		5.8	20.1	2448.3	4084.5		
2616.9	3821.3					25.5	25.5
		2.2		2433.1	4110.0		
2615.4	3823.5					54.9	2×27.5
		2.5		2401.0	4164.9		
2613.7	3826.0					6.1	
		3.4		2397.5	4171.0		
2611.4	3829.4						
		21.5	21.5	2318.4	4313.3		
2596.8	3850.9					18.1	
		8.0		2308.7	4331.4		28.5
2591.4	3858.9					10.4	
		9.3	21.0	2303.2	4341.8		
2585.2	3868.2					9.8	
		3.7		2298.0	4351.6		
2582.7	3871.9					25.3	25.3
				2290.5	4376.9		
2512.2	3980.6					13.9	
		26.0	26.0	2277.5	4390.8		29.9
2495.9	4006.6					16.0	
		28.8	28.8	2269.2	4406.8		
2478.1	4035.4						
		10.6					
2471.6	4046.0		22.5				
		11.9					

SERIES OF BANDS

$\frac{1}{\lambda}\times 10^7$	Difference	$\frac{1}{\lambda}\times 10^7$	Difference	$\frac{1}{\lambda}\times 10^7$	Difference
I		II		III	
3701.6					
	23.1	3712.6			
3724.7			23.3		
	21.3	3735.9			
3746.0			23.3		
	22.9	3759.2			
3768.9			24.4	3777.7	
	21.9	3783.6			20.2
3790.8			22.3	3797.9	
	21.5	3805.9			23.4
3812.3			20.1	3821.3	
		3826.0			
			24.9		
		3850.9			
			21.0		
		3871.9			

LENGTH OF COLUMN OF GAS = 20 CM

AT PRESSURE OF 1.35 CM

λ	$\frac{1}{\lambda} \times 10^7$	Successive Differences	Group Differences
3178.6	3146.4		
		28.2	28.2
3150.-	3174.6		
		21.0	21.0
3129.3	3195.6		
		4.8	
3124.6	3200.4		
		20.5	20.5
3104.7	3220.9		
		12.0	
3093.2	3232.9		
		3.7	
3089.7	3236.6		21.4
		3.6	
3086.2	3240.2		
		2.1	
3084.2	3242.3		
		1.9	
3082.4	3244.2		22.0
		20.1	
3063.4	3264.3		
		22.4	22.4
3042.6	3286.7		
		22.7	22.7
3021.7	3309.4		
		22.4	22.4
3001.4	3331.8		
		22.2	22.2
2981.5	3354.0		
		22.1	22.1
2962.0	3376.1		
		21.8	21.8
2943.0	3397.9		

λ	$\frac{1}{\lambda} \times 10^7$	Successive Differences	Group Differences
2692.3	3714.3		
		11.9	
2683.7	3726.2		22.2
		10.3	
2676.3	3736.5		
		10.8	
2668.6	3747.3		23.7
		12.9	
2659.4	3760.2		
		8.8	
2653.2	3769.0		
		9.4	23.5
2646.6	3778.4		
		5.3	
2642.9	3783.7		
		8.1	
2637.3	3791.8		
		6.6	22.8
2632.7	3798.4		
		8.1	
2627.1	3806.5		
		8.5	
2621.2	3815.0		19.3
		10.8	
2613.8	3825.8		
		26.0	26.0
2596.2	3851.8		
		8.0	
2590.8	3859.8		18.9
		10.9	
2583.5	3870.7		
2496.2	4006.1		
		30.1	
2477.6	4036.2		
		10.1	
2471.4	4046.3		
		27.9	
2454.5	4074.2		
		10.9	
2447.9	4085.1		
2298.4	4350.9		
		38.1	
2278.4	4389.0		
		38.5	
2258.6	4427.5		

SERIES OF BANDS

$\frac{1}{\lambda} \times 10^7$	Difference	$\frac{1}{\lambda} \times 10^7$	Difference	$\frac{1}{\lambda} \times 10^7$	Difference
3714.3					
	22.2	3726.2			
3736.5			21.1		
	23.7	3747.3			
3760.2			21.7		
	23.5	3769.0			
3783.7			22.8	3778.4	
	22.8	3791.8			20.0
3806.5			23.2	3798.4	
		3815.0			27.4
				3825.8	
					26.0
				3851.8	

the subdivisions of that table. These differences change gradually with the wave-length; they decrease from the longer to the shorter wave-lengths until the wide band is reached, then increase on the other side of it. The direction of this change corresponds with the change in absorption as the pressure is reduced; the bands decrease in intensity and eventually disappear first in the longer wave-lengths on the less refracted side of the wide band, and at the same time in the shorter wave-lengths on the more refracted side.

Region	Mean Difference in Frequency
380 to 350 $\mu\mu$	34
350 to 330 $\mu\mu$	25
330 to 313 $\mu\mu$	20
318 to 298 $\mu\mu$	21
272 to 258 $\mu\mu$	22
250 to 230 $\mu\mu$	30

Photographs taken with the column of gas 20 cm long show a rather more regular structure of bands; also some groups or series.

These groups of bands or series, combined with the breaking up of the wide continuous band into a considerable number of narrow bands as the pressure is reduced, suggest the possibility that all these bands may eventually be found to consist of very narrow bands or lines.

V. CONCLUSION

Although the series are at least in some cases incomplete and the differences in the wave-numbers are not equal, yet the near approach of these differences to equality with one another cannot be ignored. Thus this spectrum appears to consist of series of bands which follow approximately a law of equal differences. With the gas under conditions of pressure and temperature other than those tried, it may be found that its spectrum consists of quite definite series which follow closely a law of equal differences between the wave-numbers.

II. THE BAND EMISSION SPECTRUM OF SULPHUR DIOXIDE

I. APPARATUS AND METHOD

The usual conditions necessary to maintain the gas as a compound while under the electrical discharge were obtained as follows: The feeble electrical excitation was obtained from the secondary of a ten-inch induction coil, the primary of which was supplied with the current from three storage cells; the terminals of the secondary were placed too far apart for a spark to pass between them, and the vibrator was adjusted loosely. The spectrum tube had outside electrodes of lead foil with a layer of mica between them and the glass walls. As a further aid in preventing the decomposition of the gas, electrolytically prepared oxygen was mixed with the sulphur dioxide in the spectrum tube.

The sulphur dioxide was obtained from liquid sulphur dioxide which had been redistilled. The gas was dried by passing it through a bulb closely packed with phosphorus pentoxide; it was then admitted to the apparatus through a barometer column. Interposed between the barometer column and the spectrum tube were two U-tubes, one containing gold foil to absorb mercury vapor, and the other packed with phosphorus pentoxide to insure more perfect dryness of the gas. Similar tubes were interposed between the other end of the spectrum tube and the McLeod gauge and vacuum pump. All connections of the apparatus from the barometer column to the far side of the pump were either blown glass joints or mercury seal joints.

The spectrum tube was cleaned by soaking it in chromic acid

for ten or twelve hours, then washing it with distilled water, then with nitric acid, and again with distilled water; it was then dried by keeping it at a temperature of from 110° to 120° C. for eight or ten hours, meanwhile drawing a current of dry air through it. No bands from carbon compounds nor any of the strong hydrogen lines were ever seen or found on a photographic plate. The tube was exhausted, sparked, filled with oxygen, re-exhausted, and the process repeated until no air lines appeared.

The oxygen was prepared electrolytically from a 20 per cent. solution of crystallized phosphoric acid and dried by passing it through two bulbs loosely packed with phosphorus pentoxide and then through a U-tube closely packed with the same and plugs of glass wool. The line spectrum of oxygen was photographed in the region between λ 327 and 432 $\mu\mu$. It is well known that oxygen has no bands in this region.

As a standard spectrum that of the iron spark was used with the wave-lengths published by Kayser in his *Handbuch der Spectroscopie*, Vol. I.

The spectral apparatus was a Rowland concave grating mounted on the Rowland plan; it has a radius of 180 cm (5 ft. 11 in.), 15,028 lines to the inch, and a ruled surface 52 mm ($2\frac{7}{16}$ in.) wide. The wave-lengths were determined in the usual way by measurements made by means of the dividing engine mentioned above; the reduction factor was approximately 9.3 tenth-meters to 1 mm.

II. RESULTS

The chief difficulty in obtaining the spectrum of the compound lay in the extreme faintness of the light and the long exposures necessary. With the light from the capillary used "end-on" and a slit-width of about 0.05 mm, a continuous exposure of 45 hours gave only weak bands in the ultra-violet and very weak bands in the violet; the lines of the bands were too coarse for measurement. The wave-lengths of the heads of the bands obtained from this photograph are given in Table IV. A continuous exposure of 69 hours with a slit-width of 0.035 mm approximately gave bands too faint for measurement. A continuous exposure of 91 hours with a slit-width of 0.018 mm approximately was spoiled by ham-

mering in another room in the building. For the spectrum for which measurements are given the pressure of sulphur dioxide was 0.27 cm, and the pressure of oxygen 0.28 cm making a total pressure of 0.55 cm at the beginning of the exposure. During exposure, oxygen and sulphur dioxide were added to try to keep the condition of the tube constant; the total pressure at the end of the exposure was 0.45 cm. Hitherto no attempt has been made

TABLE IV

BAND EMISSION SPECTRUM OF SULPHUR DIOXIDE

WAVE-LENGTHS OF THE HEADS OF THE BANDS AND THEIR RECIPROCALS

	λ	Intensity and Character	$\frac{1}{\lambda}\times 10^7$	Successive Differences
1	3271.4	4	3056.8	
				101.5
2	3383.7	5	2955.3	
				38.2
3	3428.0	4	2917.1	
				61.9
4	3502.3	5	2855.2	
				37.3
5	3548.7	5	2817.9	
				61.6
6	3628.0	4	2756.3	
				35.9
7	3675.9	5	2720.4	
				36.9
8	3726.5	2	2683.5	
				25.0
9	3761.5	f	2658.5	
				34.7
10	3811.3	5	2623.8	
				35.0
11	3862.7	2	2588.8	
				60.1
12	3954.6	5 h.	2528.7	
				33.5
13	4007.6	4 h.	2495.2	
				20.3
14	4040.6	3 v.i d.	2474.9	
				40.2
15	4107.3	2 i.d.	2434.7	
				22.4
16	4145.5	2	2412.3	
				4.6
17	4153.4	2	2407.7	
				5.6
18	4163.0	3 h.	2402.1	

TABLE V

SERIES OF BANDS

$\frac{1}{\lambda}\times 10^7$	Difference	$\frac{1}{\lambda}\times 10^7$	Difference	$\frac{1}{\lambda}\times 10^7$	Difference
I		II		III	
1) 3056.8					
	101.5	3) 2917.1			
2) 2955.3			99.2		
	100.1	5) 2817.9			
4) 2855.2			97.5		
	98.9	7) 2720.4			
6) 2756.3			96.6	8) 2683.5	
	97.8	10) 2623.8			94.7
9) 2658.5			95.1	11) 2588.8	
		12) 2528.7			93.6
			94.0	13) 2495.2	
		15) 2434.7			93.1
				18) 2402.1	

to photograph this spectrum in regions of greater wave-length where longer exposures would be necessary.

The spectrum thus obtained consists of bands with distinct heads turned toward the ultra-violet, and is thus characteristically different from the "band" or compound line spectrum of elementary sulphur obtained by Eder and Valenta, and published in their paper, "Die Spectren des Schwefels," 1898.

The reciprocals of the wave-lengths (Table IV) show that the bands can be arranged in three series with decreasing difference of wave-numbers (Table V). The series will be seen to follow roughly Deslandres' law. These series are doubtless incomplete, as the range photographed covers only 105 $\mu\mu$.

My investigation of the spectrum of sulphur dioxide was begun originally at the suggestion of Professor A. Stanley Mackenzie with the purpose of comparing it with that of sulphur as determined by Runge and Paschen and by Eder and Valenta. Owing to various causes, this plan was not carried into effect, and the research as described in the preceding pages was proposed by Professor J. S. Ames, to whom I desire to express my indebtedness and my gratitude for his kindness in directing my work and for his invaluable suggestions in the carrying out of this investigation. I wish at the same time to express my thanks to Dr. H. W. Springsteen, asso-

ciate in physics, for the generosity with which he has provided me with the laboratory apparatus needed for this piece of work; also to Dr. W. B. Huff, professor of physics, and to Dr. E. P. Kohler, professor of chemistry, for frequent help and encouragement, particularly during the early part of my work.

Bryn Mawr College,
March 31, 1906.

A STUDY OF THE ELECTRIC SPARK IN A MAGNETIC FIELD

By HELEN E. SCHAEFFER

A study of the deflection which magnetic and electrostatic fields produce upon the electric discharge at low pressures has given a clue to the nature of the particles concerned in the discharge, and has made possible measurements of their velocity and of the ratio of their charge to their mass. A similar study of the spark-discharge at atmospheric pressure has not been published, and it is the purpose of the present paper to present the results of such a study. The investigation has been undertaken with the idea that if a proper combination of conditions could be secured, a magnetic field might cause a deflection of such a character and magnitude as to separate the constituent parts of the spark, thus securing, as it were, a differentiation in space. Moreover, since the reflection from a mirror in rapid rotation indicates the time-changes which occur, the image thus obtained of the spark dispersed in a magnetic field would show the twofold separation of time and of space, and it was hoped that this separation might lead to a more detailed acquaintance with the mechanism of the electric spark.

Professor J. J. Thomson in his book, *The Conduction of Electricity through Gases*, makes on p. 522 the following statement:

> The effects produced by a magnetic field upon the spark at atmospheric pressure are very slight, although the halo of luminous gas which surrounds the course of the sparks when a number of sparks follow each other in rapid succession is drawn out into a broad band by the magnetic field.

He also states (on the same page) that Precht has found an effect of the magnetic field upon the spark at atmospheric pressure, if the spark terminals consist of a sharp point and a blunt wire; but Precht[1] has described the character of this deflection only so far as to say that its direction agrees with the electro-dynamic laws. Precht's paper is mainly concerned with a study of the different conditions in which one form of the discharge—spark, brush, or glow—becomes

[1] *Annalen der Physik*, **66**, 676, 1898.

changed into another, and of the changes in the potential difference of the terminals occurring under these different conditions.

The method of using a rapidly rotating mirror to show the separate oscillations which occur in the spark when a condenser is placed in the discharge circuit, was first employed by Feddersen,[1] who made use of it successfully to measure the period of the oscillatory discharge, and thus confirmed the theoretical work of William Thomson (Lord Kelvin). Following him many others have used it for similar measurements.

The use of a rotating mirror and later of a rotating film to gain an insight into the constitution of the electric spark was first made by Schuster and Hemsalech,[2] later by Schenck.[3] A glance at Plate XIV, Fig. 1, showing the appearance of the oscillatory spark when viewed in a rapidly rotating mirror, will make clear the summary of their combined results. The three general features of this discharge as given by Schenck are:

1. A brilliant white straight line due to the first discharge, which is sometimes followed by one or two similar weaker straight lines at intervals of half the complete period of the condenser.

2. Curved lines of light, which shoot out from the poles toward the center of the spark-gap with a velocity constantly diminishing as they move away from the poles. It will be noticed that, as the light advances from one pole, the light moving away from the opposite pole is either very weak or absent altogether.

3. A rather faint light, generally of a different color from the curved lines of light, which fills up the spark-gap and persists for a certain length of time, especially in the center of the spark-gap, after the oscillations die out.

Schuster and Hemsalech (*loc. cit.*) first found that sufficient self-induction in the discharge circuit causes the air lines to disappear from the spectrum of the spark. Later Hemsalech[4] discovered that when the self-induction is increased, the so-called spark lines disappear, whereas the arc lines in the spectrum of the spark become brighter. Schenck (*loc. cit.*) in turn made this difference the basis of a division of the lines of the spark spectrum into three groups, this division occupying the first part of his paper. These several experimental results as arrived at by Schuster, Hemsalech, and Schenck have all been noted during the present investigation, though their

[1] *Pogg. Ann.*, **116**, 132, 1862.

[2] *Phil. Trans.*, **193**, A, 189, 1900.

[3] *Astrophysical Journal*, **14**, 116, 1901.

[4] *Comptes Rendus*, **129**, 285, 1899.

bearing upon the problem here proposed is less immediate than that of other observations made by them.

The results which have a more intimate bearing here do not relate to the disappearance of the air, spark, and arc lines under certain conditions, but to what is true of them under all conditions. Since the photographs of the spectrum lines taken upon a rapidly rotating film showed the air lines to be entirely absent in all the spectra except that of the initial discharge, Schuster and Hemsalech concluded that only the initial discharge passed through the air.

By a similar method of studying the spectrum lines—the only variation being the use of a rotating mirror instead of a rotating film—Schenck brought out an interesting difference between the spark and arc lines, viz., that the spark lines appear sharply beaded to the end of the line, whereas the arc lines show only indistinct traces of beading, which do not extend to the end of the line. In other words, he concluded that the spark lines are due entirely to oscillations, while the arc lines are due partly to the oscillations and partly to something else which retains its luminosity after the oscillations cease. The spark lines are in the spectrum of the streamers which are described as the second feature of the spark; the arc lines in that of the vapor already mentioned as the third feature.

Furthermore, Schenck (*loc. cit.*) has found that the streamers emanate from the cathode and he has concluded that they do not carry the current. This view is supported by Hemsalech,[1] who, after identifying the streamers with the metallic vapor, advances the theory that the electric charge is not carried by the metallic vapor, but by the nitrogen. As addenda to his paper, Schenck gives the results of his investigation of the effect of a strong magnetic field upon the spark, the investigation having been concerned with the spark in a magnetic field of 10,000 units, both with and without the help of the rotating mirror, though his account as published includes only one feature of the change produced by the magnetic field. I quote from this account:

With no magnetic field the spark lines and the arc lines extended clear across the gap. With the magnetic field the spark line of magnesium λ 4481 extended outward from each pole only about one-quarter of the way across the gap, leaving

[1] *Comptes Rendus*, **140**, 1103, 1905.

the center free from light of this wave-length, while the arc triplet at λ 5200 extended clear across as it did without the field. When examined with the mirror revolving, the line λ 4481 was broken up into a series of short streamers separated by intervals of darkness, while the arc triplet λ 5200 was in the form of a luminosity which advanced slowly (with a velocity not greater than 0.5×10^4 cm per second) toward the center of the spark-gap being crossed by a series of streamers. The noise of the spark was increased by the magnetic field.

It will be noticed that this description of the image given by the rotating mirror when the spark is in the magnetic field, is not essentially different from that given when the spark is out of the field. Other results relating to the disappearance of the spark lines under certain conditions, though in themselves of minor importance, are necessary to an understanding of conclusions applied by Walter[1] to the results of Schuster and Hemsalech, and involved in the discussion of the present paper. Walter has shown that if the self-induction in the discharge circuit having as spark terminals an alloy of zinc and copper, is increased, the disappearance of the spark lines of zinc before those of copper cannot be explained by the fact that the melting-point of zinc is lower that than of copper, an explanation suggested by Kowalski and Huber[2] in connection with their results. The basis of Walter's objection lies in the fact that under similar conditions he found the spark lines of lead to persist longer than those of copper; whereas, if the difference in the melting-points were the determining factor, the spark lines of lead should, like those of zinc, disappear before the spark lines of copper, since the melting-point of lead is also lower than that of copper. Accordingly Walter,[3] referring to a conclusion reached in one of his earlier investigations, viz., that the metallic vapor in the spark is formed at the negative pole, is led to decide that the metallic vapor must be a result of the disintegration of the cathode. He therefore thinks that the amount of disintegration which occurs at the cathode may be the important factor in determining which lines shall persist longest when the self-induction in the discharge circuit is increased, and he finds that the lines of that metal which suffers most disintegration at the cathode persist longest.

1 *Annalen der Physik*, **21**, 223, 1906.

2 *Comptes Rendus*, **142**, 994, 1906.

3 *Boltzmann-Festschrift*, 647, 1904.

This conclusion, together with Schenck's observation that the spark lines are affected by the magnetic field, while the arc lines are not, Walter considers a sufficient explanation of the differences which Schenck and Hemsalech have observed in the behavior of the spark lines and arc lines. The metallic particles torn from the cathode by disintegration he thinks carry with them an electric charge which they do not lose until they have reached the center of the spark-gap. The spark lines are characteristic of the light from the metallic particles which carry an electric charge; the arc lines, of that from the metallic particles which have lost their charge.

To explain Hemsalech's result that increase of self-induction causes the spark lines to disappear from the spectrum and the arc lines to become brighter, he says that increase of self-induction lengthens the period of oscillation and decreases the current in the single oscillations. This decrease of current causes a longer interval to elapse between disintegration and luminescence of the particles, thus giving time for a greater number of particles to lose their charge. With added self-induction the ratio of the uncharged particles to those charged increases. Therefore the arc lines characteristic of the uncharged particles are brighter than the spark lines characteristic of the charged particles.

Returning to the results of Schuster and Hemsalech, we find that by means of the curvature which a rotation of the photographic film produces in the metallic spectrum lines, they have obtained as the magnitude of the velocity of the particles of many different metals a value of 4×10^4 cm per second. Schenck, on the other hand, obtaining a value of 25×10^4 cm per second, is led to believe that the difference between his values and those of Schuster and Hemsalech may be due to the fact that they measured the slope of the locus of the extremities of the streamers, while he measured the slope of the streamer itself.

The present investigation may be divided into three parts:

I. A study of the visible space-changes which the presence of a strong magnetic field causes in the spark.

II. A spectroscopic analysis of the different parts into which the spark is spread out under the influence of the magnetic field, this analysis being made solely for purposes of identification.

III. A study of the image of the spark given by a rapidly rotating mirror when the spark is in a magnetic field. The object of this part of the experiment was to get a second differentiation of the spark, viz., a differentiation with respect to time of the space-changes described in Part I.

Three types of electric spark were studied in each of the three parts of this investigation.

1. The spark obtained when neither capacity nor self-induction has been introduced into the secondary circuit of the induction coil.

2. The spark obtained when a capacity of 0.0005 to 0.012 microfarads has been introduced into the secondary circuit.

3. The spark obtained when a capacity of 0.0005 to 0.012 mf and a self-induction of 0.003 henries have been introduced into the secondary circuit.

APPARATUS

The spark was obtained from an induction coil, the primary of which was supplied by a direct current of one to four amperes taken from the 110-volt mains. The potential of the secondary could be raised high enough to produce a 32-cm spark between its poles. With a capacity of 0.012 mf and a self-induction of 0.003 henries in the secondary circuit, a spark of about 2 cm length passed between the metallic terminals. The capacity was obtained from Leyden jars arranged in parallel in the secondary circuit and was varied by gradually changing from a $\frac{1}{4}$-gal. jar to six 1-gal. jars, each 1-gal. jar giving a capacity of about 0.002 microfarads. The self-induction was obtained by placing in the secondary circuit four wire spools arranged in series. An adjustable resistance in the primary circuit served to change the spark from a very noisy to a hissing one. An approximately uniform magnetic field was obtained over a distance of about 2 cm by using truncated cones as the pole-pieces of a large DuBois electro-magnet.

The spectroscopic analysis was made by visual observations and photographs. The former were made by means of a calibrated prism-spectroscope which was mounted upon a carriage that could be moved at right angles to the rays of light falling upon the slit of the spectroscope. In this manner the spectra given by the different parts of the spark could be conveniently studied. The spectrograms

were made by means of a Fuess quartz-prism-spectrograph with camera attached.

For the third part of the investigation the image of the spark was reflected from a plane metallic mirror made by Brashear. This mirror was 5 cm in diameter and was mounted so that its rotation was about a horizontal axis. It was driven by a means of an electric motor and could be rotated at a speed of 200 revolutions per second, although a speed of about 50 revolutions per second usually sufficed. The speed of the mirror was measured by the impressions which a bristle attached to its axis made upon a revolving drum. These impressions showed that after the mirror had been in rotation for a short time its speed was practically constant and even such deviations from its constant value as occurred were found to be well within the limit of experimental error.

I. EFFECT OF THE MAGNETIC FIELD

The deflection produced by the magnetic field is most striking when the spark is allowed to pass along the lines of magnetic force or perpendicular to them, the deflection taking the form of circles in the latter case and of spirals in the former. (See Plate XIV, Figs. 3 and 2.) The spirals seem to be wound about cones of revolution. having different angles of divergence, whereas the circles all lie in a plane perpendicular to the lines of magnetic force. In a magnetic field of 1050 units the central threads do not participate in this spiral or circular form.

To appreciate in full detail this effect of the magnetic field upon the spark, a description of the three types of spark-discharge, as they appear both in and out of the field, will be necessary. The first type consists of one or two reddish-white threads which pass directly across the gap and are accompanied by a reddish, luminous vapor that assumes a yellow tinge when the current through the primary circuit is increased. Without the magnetic field this vapor forms an envelope about the central threads: in a parallel field it is deflected into a spiral sheet; in a transverse field into two semicircular sheets which are in the same plane. If the current through the primary circuit is sufficiently small, there is only one such spiral sheet in the first case, and only one plane semicircular sheet in the

second. If the current is increased, two spiral sheets or two semicircular ones are present and the latter two are in the same plane, one of them being on either side of the spark-gap. If, however, the spark terminals are drawn sufficiently far apart, one of the two spiral or semicircular sheets disappears entirely. In a field of 12,000 units, however—the strongest that could be obtained with the amount of current available, viz., 19 amperes—no deflection of the central threads could be noticed in either position of the magnetic field.

It was found that a small capacity in the discharge circuit introduced several reddish-white threads into the semicircular and spiral sheets. These threads took the form of spirals in a parallel field, the form of semicircles in a transverse field, and all lay in a single plane perpendicular to the lines of magnetic force. A slightly larger capacity in the discharge circuit made these threads more brilliant and increased their number. Strengthening the magnetic field also increased the number of these threads and their brilliancy. An increase in magnetic field-strength seems therefore to produce the same effect as an increase in capacity.

The second type of spark consisted of a bundle of very brilliant white threads, which were accompanied by little or no vapor. With a capacity greater than 0.002 mf this vapor was not present. In the magnetic field it assumed a circular or spiral form, according to the position of the spark-gap, and was accompanied by thin, brilliant white threads, which likewise were parts of circles or spirals. (Plate XIV, Fig. 5.) This vapor was yellowish in color, whatever terminals were used, and was spread out into a sheet that was so thin as to be almost invisible. The bundle of threads across the gap could not be changed by any available strength of field.

Plate XIV, Fig. 4, shows the central threads and metallic vapor of the third type of spark as they appear without the magnetic field. The threads are not so brilliant as those of the second type of spark, and have the same reddish color for all the metals tried as terminals. The color of this metallic vapor, however, varies with the metal used as spark terminal. With aluminium it is a bright green and shoots out from the electrode instead of enveloping it. With magnesium this vapor is yellow-green; with calcium, pink; with zinc, cadmium, and lead, orange-red with a blue core extending a short distance

from the electrode. This vapor does not appear to advance farther from the electrodes as the spark length is increased. It is therefore possible to separate the poles to such an extent that this vapor seems to be entirely absent from the center of the spark. With a given capacity in circuit the length of this vapor increases, however, with an increase of self-induction.

The figure just mentioned was taken from a photographic plate which was not sensitive to the reddish-yellow vapor enveloping the brilliant threads when no magnetic field is present. If the spark terminals are sufficiently far apart or the current through the primary is small this vapor is entirely absent. In the presence of the magnetic field it is changed to bright threads unless the spark terminals are close together. These threads are parts of circles or spirals according to the position of the spark terminals in the magnetic field. Strengthening the magnetic field increases their curvature. The color of these threads varies with the metals used as spark terminals. With aluminium they are reddish-white; with magnesium, red; with calcium, blue; with cadmium, reddish-purple; with zinc, lead, and bismuth, reddish-white. As the capacity and therefore the period is increased, these threads become broader, fewer in number, more red in color—where aluminium is concerned—and tend to depart from the plane perpendicular to the lines of magnetic force in a transverse field. Changing the amount of self-induction in the circuit does not seem to introduce any change into the form or number of the threads; but, if with a capacity of 0.002 mf in a circuit the self-induction is entirely removed, the threads are brilliant white instead of being reddish in color, and they disappear from the immediate region of the central threads, thus decreasing their number considerably. (Compare, Plate XIV, Figs. 7 and 5.) When self-induction is present, these threads which take the form of circles or spirals, according to the position of the spark-gap, can be obtained with a capacity as great as 0.012 mf. Without self-induction it is impossible to obtain any spiral or circular threads with a capacity greater than 0.002 mf. The number of these threads present when the third type of spark is in a magnetic field, passes through a maximum as the capacity is increased from 0.0005 to 0.012 mf, this maximum number occurring when the capacity is about 0.002 mf.

The number of threads present in the second type of spark also passes through a maximum, but here the maximum number occurs when the capacity has a much smaller value, comparable with that obtained from a small parallel-plate condenser. On the other hand, the width of the threads in both types of spark does not pass through a maximum for the range of capacities used, but steadily increases as the capacity is increased. If the electrodes are so far apart that without the magnetic field no vapor envelops the central threads, none of these circular or spiral threads is present when the magnetic field is on. It the electrodes are close together, the vapor is spread by the field into a yellow, circular or spiral sheet, instead of being broken up into brilliant circular or spiral threads.

It requires a much stronger field, however (about 12,000 as compared with 1050), to secure a noticeable change in the central threads. They are twisted by a very strong field along the spark length into a spiral, much like the thread of a screw, and of small radius. (See Plate XIV, Fig. 6.) A field at right angles to the spark length seems to cause a very slight general curvature in these central threads and also to make them appear crenate, the whole being concave to the spark-gap. The number of spiral turns or small semicircles does not in the latter case remain constant, and this irregularity suggests that these spirals or semicircles may be brought about by a sudden change in the velocity of the particles resulting from a loss or gain of electrons.

With this field of 12,000, the metallic vapor of the third type of spark also undergoes a deflection. In a transverse field it certainly assumes a circular form, but in one that is parallel, its form though much changed is too indistinct to be called that of a spiral.

The results thus obtained when any of the three types of spark is in a magnetic field are interesting if compared with the results obtained by Wehnelt,[1] when a hot lime cathode was used for the discharge at low pressures. He found that the particles emitted by a hot lime cathode bring to luminescence the gas through which they pass, and this luminescence indicates the spiral, or circular paths in which charged particles under the influence of a parallel or transverse magnetic field have been shown theoretically to move.

[1] *Annalen der Physik*, 14, 425, 1904.

The present investigation seems to show that also at atmospheric pressure there are particles which describe luminous paths in the form of spirals and circles. A much stronger field is necessary to produce the deflection here than at low pressures and the radii of the spirals and circles are much smaller.

These observations seem to justify two conclusions, at least, as regards those particles with which luminescence in the spark at atmospheric pressure is associated:

1. They obey in general the laws of motion which experiment and theory have shown charged particles to obey when at low pressure and under the influence of a magnetic field.

2. The obedience to these laws certainly lends strong support to the view that the particles carry an electric charge.

For two reasons it has at present seemed impracticable to find out whether the curvature of the path of these particles, as given by actual measurement, satisfies an equation deduced from theoretical considerations. The electrical conditions are seriously complicated by the necessity of having the electrodes sufficiently close for the passage of a spark at atmospheric pressure, and it would therefore be difficult to find the true values and directions of the electrical forces. The mathematical theory of the behavior of charged particles in a magnetic field has been worked out only in a general way for atmospheric pressure.

Figs. 7 and 8 show a twofold asymmetry in the deflection produced by a magnetic field: (1) an asymmetry at the electrode itself; (2) an asymmetry in the width of the two semicircular, luminous sheets. The latter asymmetry will be considered first.

Figs. 7, 8, 10, and 11 show the difference in the width of the two semicircular sheets. This difference also seemed to exist in the two spiral sheets, but their position as well as their spiral form made it more difficult to compare their respective widths. When the direction of the current through the primary, or that of the magnetic field is reversed these two sheets exchange places (compare Figs. 11 and 10 with Fig. 8). Furthermore when the current through the primary is decreased, or the distance between the spark terminals is increased, both of the sheets become steadily narrower, until finally a stage is reached where only one of the two is present.

The difference in the width of the two sheets can hardly be explained by the fact that the magnetic field may not have been entirely uniform throughout the region of the spark, since this difference in width was found to persist even in that part of the field which was far from uniform. An explanation might be sought in the fact that on one side of the spark-gap the magnetic field, due to the passage of the current, reinforces the permanent field given by the electro-magnet, while on the other side of the spark-gap, it weakens the permanent field. This explanation, however, would require both sheets to be produced at the same time and this simultaneous passage seems improbable since both sheets are later found to be due to particles of like charge. Fig. 9 seems to indicate that the two sheets are not formed at the same time. This photograph was taken when both sheets appeared to be present; yet it shows only one sheet of threads, thus suggesting that the exposure ($\frac{1}{50}$ sec.) was short enough for the set of threads on one side of the spark-gap to be photographed before that on the other was formed.

At the spark terminals the ends of the sheet are asymmetric in the following respect. One end of the semicircular boundary rests on the point of the electrode, while the other end of the boundary is at some distance from the point of the opposite electrode, as may be seen in Figs. 8, 10, and 11.

It has already been stated that the image given by the rotating mirror shows the particles with which these luminous sheets are connected to be most probably negative. If they are negative, then the direction of the field, together with the direction of the deflection, shows that they *advance* from the point of the electrode and *end* in a straight line extending for some distance along the other electrode.

In terms of the brilliant circular threads, characteristic of the spark which results when both capacity and self-induction are inserted into the secondary circuit, this asymmetric form at the electrode may be described thus: The threads all proceed from the extreme end of the negative electrode and end at different points on the positive electrode, these different points being in a straight line and all lying in the plane of the sheet which is perpendicular to the lines of magnetic force. (See Plate XIV, Fig. 7.)

Actual measurement has shown that these circular threads do not

possess the same radius of curvature. The asymmetry at the spark terminal may, accordingly, have no deeper significance than the fact that the circular threads are compelled to end upon different points of the positive terminal because their curvature is different, whereas they all start from the same point of the opposite, negative terminal because its potential is higher than that of any other part.

It has already been noticed, too, that if the conical end of the metal terminal is not perfectly smooth, the sheet sometimes starts from one or two other points in addition to the extreme point of the spark terminal. These few points were very different, however, from the line of points in which the sheet ended on the opposite spark terminal—that line of points lying, together with the axis of the spark terminal, in a plane which was at right angles to the magnetic field. Except at the few points from which the sheet seemed to proceed, a space could be seen between the sheet and the terminal from which the particles appeared to start. At the opposite, positive terminal no such separation could be seen between any part of the sheet and the spark terminal. When, therefore, the sheets seemed to start also from other points of the negative terminal, it was supposed due to the fact that an unevenness of the surface of the terminal caused these points to act as additional centers of discharge.

Fig. 10 shows the change which occurs in the position of the sheets when the direction used above for the magnetic field is reversed. It will be seen that the two sheets interchange sides as though each were turned through an angle of 180° about an axis along the spark length.

Fig. 11 shows another difference in the position of sheets, occurring when the current through the primary is reversed, the sheets here undergoing what might be termed a diagonal inversion. Not only does each sheet turn through an angle of 180° about an axis along the spark length, but each end turns, as it were, through another angle of 180° about an axis perpendicular to the spark length and in the plane of the sheet.

The first type of inversion is in entire agreement with such a change in the deflection, as a moving charged particle would experience in a magnetic field in which the direction has been reversed. The second type is also what the electro-dynamic laws would lead

one to expect for reversal of current through the primary of the induction coil.

II. A SPECTROSCOPIC ANALYSIS OF THE DIFFERENT PARTS OF THE SPARK

An image of the semicircular, reddish sheet presented in a transverse magnetic field by the first type of spark was focused upon the slit of a quartz spectroscope. The slit was at right angles to the spark length so that if any differences existed in the various parts of the sheet they might be shown upon the same plate. A magnetic field sufficiently weak to keep out of the sheet all reddish, circular threads was chosen.

Three different spectrograms were made of each of the following metals: aluminium, bismuth, zinc, cadmium, and lead. The first shows the spectrum of the outer part of the semicircular sheet; the second, that of the part containing the bright threads which pass straight across from pole to pole; the third, that of the third type of spark, taken merely as a means of comparison, and for this purpose it answers very well, inasmuch as the presence of self-induction brings into prominence the metallic lines. Plates were taken with the first spectrum directly above the third; others, with the second above the third. The first two spectrograms were given exposures of one hour; the third, of a minute.

As a result of these experiments the luminous sheet was found to present the same spectrum for each metal tried. This was the spectrum of the nitrogen bands and was found to correspond with that obtained from a low-pressure discharge tube containing nitrogen. The spectrum of the bright threads across the gap showed lines, identified visually with the so-called air lines, and lines corresponding in position to the metallic lines which show prominently in the third spectrogram. These three spectra may be seen in Figs. 12 and 13.

Since the sheets here studied show only the nitrogen bands in their spectrum, it seems probable that whereas their form indicates the path of the charged particles through the air, their luminescence is merely that of the air particles and is not in any way shared by a light characteristic of the metallic terminals from which the charged particles appear to come. On the other hand, since the central

threads have the metallic lines in their spectrum, there is reason to believe that they are, in some way, associated with particles which emit a radiation characteristic of the metallic terminals; but which cannot be considered as charged until a deflection or some other evidence is obtained.

The presence of the magnetic field introduces into the intensity of the lines a difference which is interesting. It will be remembered that, with the magnetic field absent, a yellowish vapor envelops the central threads, if sufficient current passes through the primary; also that with the field present, this vapor is spread out into a plane sheet passing through the spark-gap and perpendicular to the magnetic field, thus leaving the region about the central threads free from vapor except in the plane of this sheet. If the spark in the magnetic field is viewed side-on, i. e., if the spectroscope is placed so that no luminous vapor intervenes between it and the central threads, the metallic lines are brighter than when the field is absent. On the other hand, if the spectroscope is placed with its slit in the plane of the sheet of luminous vapor, so that the width of this sheet is between it and the central threads, the metallic lines are fainter than when the vapor surrounds the central threads, as always occurs when there is no field. Since an hour's exposure showed no evidence of these metallic lines in the spectrum of the vapor of the first type of spark, it seems probable that the decrease in the intensity of the metallic lines is due merely to the passage of the light through a cloud of particles and not to any such absorption as could cause a reversal.

The difference of intensity just described has also been noticed in the brightest metallic lines shown on the spectrograms of the third type of spark. When this type of spark is viewed side-on, these lines seem at least twice as bright in the field as out of it.

By extending to the second type of spark the spectroscopic analysis made for purposes of identification, it was found that the bundle of brilliant white threads which pass directly across from pole to pole and are undeflected by any available field have the well-known spectrum which presents itself when capacity is introduced into the secondary circuit, a spectrum consisting of bright air lines and fainter metallic lines. The spectrum lines of the circular threads

have the same wave-lengths as those of the non-deflectable threads, but they are much fainter. Throughout this paper the word *non-deflectable* is used in a purely relative sense, viz., that the central threads could not be deflected in any available strength of field of 12,000 units. Only for this type of spark is the spectrum of the circular threads the same as that of the threads which pass directly across the spark-gap.

In the third type of spark the investigation was concerned with the spectrum of the bright circular threads, that of the brilliant white central threads, and that of the vapor which extends several mm from the electrodes. All three different spectra were studied for electrodes of aluminium, zinc, bismuth, cadmium, lead, calcium, and magnesium, and no attempt was made to measure the lines with greater accuracy than was necessary for purposes of identification. For the visible spectrum a calibrated prism-spectroscope served to identify the arc, spark, and air lines accurately enough with those given for these metals in the charts of Hagenbach and Konen. The accompanying photographs show the spectrograms of the three different parts of the spark of each metal (taken directly, one above the other, upon the same plate). Plate XIV, Fig. 14, shows the spectrogram obtained when magnesium terminals were used. Fig. 15 shows the spectra of the central threads of each metal, taken one directly above another, and all by focusing upon the slit that part of the spark which is free from the metallic vapor. The time of exposure was two minutes for the spectra of the circular threads and of the central threads; one minute for those of the vapor close to the poles. For the spectra of the central threads the spark-gap was lengthened until the center appeared free from the metallic vapor enveloping the poles. The spectrograms obtained for the seven metals used as spark terminals, together with the visual observations upon the spectra of these metals, gave in general the following results:

1. The circular threads show spectra composed of faint air lines and the bright arc lines characteristic of the metal used. The spark lines appear to be entirely absent.

2. The central threads across the gap show the spectra of the air lines. Plate XIV, Fig. 15, shows that these spectra are practically the same for all the metals used. On some of them the metallic lines

show so faintly that the suggestion is rather that of a diffuse light reflected upon the slit than that of a light coming directly from the threads themselves. This view seems especially valid if one considers that with the spark terminals closer together the metallic lines of this type of spark are very much brighter than the air lines.

3. The vapor near the poles could not be isolated from either the central or the circular threads. Accordingly spectra taken in its region near the spark terminals showed very bright arc and spark lines together with very much fainter air lines. The other two spectra described above in 1 and 2 do not present the spark lines showing these spectra.

Varying the capacity or the self-induction changed only the intensities of the spectrum lines.

III. STUDY OF THE SPARK PLACED IN A MAGNETIC FIELD AND REFLECTED FROM A MIRROR IN RAPID ROTATION

To measure the velocity of the streamers, Schenck, and Schuster and Hemsalech used a method based upon a measurement of the slope of the streamer as given by the mirror in rapid rotation. As the luminescent vapor advanced from the spark terminal toward the center of the spark-gap, the light from this vapor reflected from the mirror when stationary and focused upon the photographic plate described a straight, horizontal line, a true representation of the path of the vapor. When the mirror is in rotation, however, this image is drawn out in a direction perpendicular to that in which the vapor is advancing. The resultant path on the plate is curved because the velocity of the vapor decreases as it approaches the center of the spark-gap.

This method, based upon a measurement of the apparent change of form introduced by the rotation of the mirror, would involve serious complications if it were used to measure the velocity of the particles from which arise the brilliant, circular threads of the spark of the third type. Measurement shows that the curvatures of the threads in a single spark vary considerably among themselves. Accordingly, even if two images of the same single spark-discharge were obtained—the one with the mirror in rapid rotation, the other with it at rest—it would be very difficult to match the threads in the

two images and then to measure the change of curvature introduced by the mirror. The existence of such a curvature change, however, suggests a simple method of measuring the velocity of the particles associated with the circular threads, and this method has been adopted in the present investigation. The method is this. The spark is made to pass in a horizontal plane parallel to the horizontal axis of the mirror. The spark terminals are so placed that with the mirror at rest the two ends of each circular thread are at exactly the same distance from the bottom of the photographic plate. When the mirror is set in rapid rotation, the image of each thread shows one end to be farther from the bottom of the plate than the other, this distance being greater for a long thread than for a short one. Evidently a time-interval must have elapsed between the formation of the two ends of the thread, and the existence of this time-interval shows that the luminosity of the circular threads must somehow be produced by the movement of a single set of particles from one pole toward the other, and not by two sets of particles which start simultaneously each from its own pole. From this time-interval may also be calculated the average velocity of particles. This average velocity is equal to the length of the circular path, divided by the time-interval above mentioned. This time-interval bears the same ratio to the time of one revolution of the mirror as that borne to 2π by the angle which is swept through in describing the distance a ($a = y_1 - y_2$, measured along the axis of y, Fig. 16) between the two ends of the thread. By means of a comparator, reading to thousandths of a millimeter, the distance a was measured upon a photographic plate which was moved parallel to the path described by the image of the spark across it. Readings to hundredths of a millimeter were found to be within the limit of experimental error. The length of the circular thread itself was measured by making a fine flexible wire coincide with an enlarged image of the thread, and then measuring the length of this wire after it had been straightened. Two errors are introduced into this latter measurement by the relative motions of the mirror and the particles. These two errors, being of opposite sign, offset each other. When the particle and the mirror are moving in the same direction, the length of the path of the moving particle is shorter in the image than it is in reality,

whereas, when they move in opposite directions, the path of the particle is longer in the image than in reality. As both these cases occur in the same semicircular thread the sum of the two errors becomes practically zero. Errors due to a displacement of the image by the rotation of the mirror were found to be well within the limit of experimental error. The spark terminals, besides being placed in such a position, were chosen of such a width and form that they could introduce no serious error into the measurement of a. The accompanying table gives the values of the velocities thus calculated from measurements upon the circular threads.

TABLE I

Number of 1-gallon Leyden Jars in Circuit	Values of the Velocity in cm per sec.
1	6.3×10^4 to 8.5×10^4
2	4.8×10^4 to 6.7×10^4
3	4.4×10^4 to 6.0×10^4
4	4.3×10^4 to 4.9×10^4
5	3.8×10^4 to 7.0×10^4
6	3.9×10^4

It is seen that the velocities are roughly of the order 5×10^4 cm per sec. Within the limit of error it cannot be said that there is any difference for threads of different curvature, nor is there a serious difference when the capacity is gradually changed from that given by one 1-gallon Leyden jar to that given by six 1-gallon Leyden jars.

This method may possibly be used to measure the velocity of the particles associated with the central threads, if the spark length be so adjusted that the threads remain in the same plane throughout their length.

This difference in the position of the two ends of a circular thread, as shown in the image reflected from the rotating mirror, also led to a determination of the sign of the charge carried by the particles whose velocity has just been calculated. According to the direction in which the mirror rotated, the end of the thread last formed was nearer or farther from the horizontal edge of the photographic plate. Thus from the direction of rotation and the position of the ends of the thread on the photographic plate, it was found which end of the thread was first formed, and this fact in turn indicated the pole from which the particle started and the direction in which it was moving. This direction, together with that of the deflection and that of the magnetic field, gave the sign of the charge carried by the particle. It was thus found that a negative charge is carried by the particles to which the easily deflected, circular threads are due.

As already stated, the equations of motion of a charged particle in a magnetic field are still, so far as atmospheric pressure is concerned, very general and incomplete. Moreover, the electrical conditions, complicated by the nearness of the electrodes required for the passage of a spark at atmospheric pressure, introduce other difficulties, so that it is not easy to arrive at an equation which will accurately represent the motion of these charged particles connected with the circular threads. The use of the equation $\frac{1}{\rho}=\frac{eH}{mv}$ in order to find the magnitude of $\frac{e}{m}$ would have no other justification than the fact that the path of these particles has approximately the same circular form as that of the particles in a low pressure discharge under similar magnetic conditions, the curvature of this form in the latter case satisfying the equation just mentioned. Some of the photographs show a change in the curvature of the threads at a distance of about 2 mm from the spark terminals. The radius of curvature then becomes smaller but has a constant value to within 2 mm of the opposite terminal, when it may become greater by as much as 5 per cent. In the present investigation

$$H=1050 \text{ c.g.s. units,}$$
$$V=5\times 10^4 \text{ cm per sec.}$$

ρ varied from 0.4 cm to 0.7 cm, as may be seen in the accom-

panying table which gives the values of ρ for different amounts of capacity in the secondary circuit. If these values are substituted in the equation $\frac{1}{\rho}=\frac{eH}{mv}$, $\frac{e}{m}$ varies from 1.2×10^2 to 0.7×10^2.

TABLE II

Number of 1-gallon Leyden Jars in Circuit	I Series of Measurements Values of ρ in cm	II Series of Measurements Values of ρ in cm
1	0.56	0.42 0.70
2	0.60	0.52 0.54 0.60 0.65
3	0.55	0.50 0.54 0.60
4	0.60 0.55 0.48	0.43 0.45 0.57 0.43
5	0.45	0.45
6	0.40	0.40 0.43

Two or more values of ρ for the same number of Leyden jars are those belonging to different threads upon the same photograph, not several values of ρ belonging to the same thread. $\left\{\begin{matrix}0.57\\0.43\end{matrix}\right.$ are the radii of curvature of different parts of the same thread, 0.57 being the ρ of the parts near the spark terminal.

Measurements were also made upon the slope of the streamers to find how the value obtained for these velocities agreed with the values obtained by Schenck, and Schuster and Hemsalech, Schenck having obtained a value of about 25×10^4 cm per sec.; Schuster and Hemsalech one of 4×10^4 cm per sec.

The measurements made here upon the streamers have shown a decrease in the velocity as the slope was measured from the electrode toward the center of the spark-gap, the values of the velocities ranging from 1×10^5 cm per sec. to 4×10^3 cm per sec. Measurements were taken only upon the part of the streamer which is not in the same direction as the path of the image across the plate.

Moreover, by closely examining the streamers it will be noticed that the second streamer advances farther toward the center of the spark-gap than the first, the third farther than the second, etc., the the brightness of each diminishing as it nears the center of the gap. The slope of each succeeding streamer becomes after a short time less abrupt than that of its predecessor and their points of junction finally lie on one continuous line, which is almost parallel to the path of the image across the photographic plate. It will also be noticed that the space between successive oscillations increases. This increase in space means that the interval of time between the oscillations becomes greater as they die out and this suggests that each streamer, before it joins the next one, approaches the center of the gap more nearly than its predecessors, for the reason that the vapor is there given a longer time to diffuse toward the center. The decrease in slope shows that the change in the velocity of the vapor becomes less abrupt with each successive oscillation, suggesting that the sum total of the forces which act upon the vapor changes less abruptly with each oscillation. This would naturally be expected from the curve of an oscillatory discharge. These observations, together with the results given on p. 141, lead one to think that Schenck may have been mistaken in suggesting —as he did, to explain the difference between his values for the velocity of the streamers and those of Schuster and Hemsalech— that they measured the slope of the *locus* of the extremities of the streamers. Such a locus is almost parallel to the path of the image of the spark across the photographic plate and a measurement of it could not possibly give for the velocity a value comparable with that secured by Schuster and Hemsalech. It seems possible therefore that the velocities measured were actually those of different parts of the streamer; Schenck having measured that of the part very near the electrode; Schuster, that of the part somewhat nearer the center of the spark-gap.

This possibility suggested that there might be for some of the metal terminals a noticeable difference in the parts of the streamer itself. With zinc, cadmium, and bismuth a difference in color was noticed. For a very short distance, not exceeding 2 mm, the streamer was of a brilliant blue color like that of the blue cone noticed in the vapor about the electrode. Then it changed to a dull blue and

afterward to an orange-red like that of the vapor at some distance from the electrode. Furthermore the color of the bright blue core is like that of the bright points of light seen where the sheet of vapor of the first type of spark just touches the metal terminals, and where the circular threads touch the terminals in the third type of spark, provided that a capacity less than 0.012 mf is present in the circuit.

Plate XIV, Fig. 17, shows the spectrum of the spark when the spark length is parallel to the slit of the spectroscope. The spark line λ 4481 of magnesium is seen to be present only in the region of the spark terminals, whereas the other lines extend entirely across the spark-gap. When other metals were used as terminals, similar plates, showing the spark lines present only in the neighborhood of the terminals, were obtained.

The photographs show that the vapor represented by the very bright part of the streamer exists for a short time in each oscillation, but does not persist until the next oscillation at that electrode has begun: it therefore does not receive a fresh addition from each successive oscillation. The rest of the vapor, on the other hand, does persist until after the second or still later oscillations have begun, and thus presents a continuous background of light, reinforced by each successive oscillation. Schenck, it will be remembered, found that the image of the spark line given by the rotating mirror was sharply beaded and that the parts of the line are separated by intervals of complete darkness. The arc lines, on the other hand, showed only indistinct traces of beading, such as would be given by a continuous background of light crossed by streamers. These two facts taken in connection with the foregoing description lead to the following inference. The bright core, entering with each oscillation and completely dying out before the next begins, has some association with the spark lines which show by their distinct beading that they arise from something ending before the next oscillation has begun. The rest of the vapor, on the other hand, bears some relation to the arc lines which, by their indistinct beading and continuous background, show that they are associated with something persisting throughout and receiving fresh additions with each successive oscillation.

By allowing the light from the spark to fall first upon a plane

grating, and then upon the rotating mirror an attempt was made to see if the spark lines extended only as far as the bright blue core and if they died out before the next oscillation. But for every metal tried, the spark lines in the visible spectrum were too close to the arc lines, and the image given by the mirror in rotation lasted too short a time to give any positive results in this connection.

This method of using a grating objectively and at the same time a mirror in rotation, did however show that the continuous spectrum is in the form of the irregular first discharge which extends across the spark-gap. (See Plate XIV, Fig. 1.) This figure also shows instead of one or two discharges as Schenck has observed (cf. p. 122) that there may be as many as six or seven discharges following the path of the first discharge.

The velocity of the streamers Schuster and Hemsalech found to be about 4×10^4 cm per sec. The order of the value obtained in the present investigation for the average velocity of the particles connected with the circular threads is 5×10^4 cm per sec.; and the close agreement between these two values led me to try to see if there were any relation between that part of the streamer measured by Schuster and Hemsalech, and the circular threads. It was thought that if an effect of the magnetic field upon the metallic vapor could be found, some relation between this vapor and the circular threads might be traced. Accordingly the oscillatory spark obtained with a capacity of 0.012 mf and a self-induction of 0.003 henries was made to pass in the strongest available magnetic field, in order to show whether the metallic vapor acts in a manner at all analogous to that characteristic of the brilliant circular threads occurring under conditions which are similar in every respect to the preceding except that less capacity is present in the discharge circuit. To obtain oscillations sufficiently separated for the study of the vapor just described a capacity of 0.012 mf was necessary, and this type of oscillatory spark showed no deflection in the magnetic field used for obtaining the circular threads. In a field of 12,000 units, however, the metallic vapor of this oscillatory spark was deflected into the form of broad, circular rings much like the circular threads, except that they were broad and not brilliant. It is possible that a much stronger field might introduce narrow, brilliant threads, just as an increase

in the strength of the field introduced threads into the sheet of vapor belonging to the first type of spark. The brilliant blue core still remained close to the spark terminal at the two ends of each broad ring of vapor. If the magnetic field had caused any change in the core, this change would be difficult to detect because of the shortness of the core.

Photographs both in and out of the magnetic field were then taken with the mirror in rotation, in order to show whether the magnetic field produced a difference in the streamers. Both the bright core and the other vapor of the streamers showed irregularities when the spark was in the magnetic field, and these irregularities were such as a curved deflection might introduce into the motion of the particles giving the streamers. This suggested that the bright core, as well as the rest of the metallic vapor, was associated with charged particles, and additional evidence for this theory was afforded by the circular form of this vapor in a very strong field. If then every luminous part of the spark is associated with charged particles, Walter's theory that the arc lines are due to particles which have lost their charge seems doubtful. In whatever part of the spark the arc lines may originate, it seems probable that they must in any case arise from a luminescence excited by charged particles, since every part of the oscillatory spark suffers some deflection in the magnetic field, and this deflection obeys the electro-dynamic laws.

These arguments taken alone are, of course, insufficient to prove that the bright core seen in the third type of spark and the bright points of light seen in the first type of spark at the terminals have as their characteristic spectrum lines the spark lines and that the vapor envelope has the arc lines; but they give a definite support to the theory. Such a theory if proved would add weight to Schenck's suggestion that the spark lines are due to peculiar vibrations arising when the atoms are torn from the metal terminals, whereas the arc lines are due to the more fundamental vibrations which persist after the abnormal vibrations have died out.

BRIEF SUMMARY OF RESULTS

The three types of spark studied are described on p. 126.

1. When the spark is placed in a magnetic field, the direction of

which is parallel to that of the spark-gap, the first type of spark presents two sheets of vapor in the form of spirals. In the field at right angles to the spark length this vapor is in the form of two semicircular sheets, one being on each side of the spark-gap in a plane perpendicular to the direction of the magnetic field.

In the second type of spark (if the capacity did not exceed 0.002 mf) and in the third type of spark brilliant spiral threads in a parallel field and brilliant circular threads in a transverse field took the place of the spiral and circular sheets respectively. In the first and second types of spark the bundle of threads across the gap could not be deflected by a magnetic field of 12,000, the strongest to be obtained with the available amount of current, viz., 19 amp. In the third type the metallic vapor and the threads across the gap were deflectable in a very strong field and in a manner analogous to that of the circular and spiral threads.

The character of the deflection seems to furnish good reason to infer that the particles with which luminosity is associated possess an electric charge. A twofold asymmetry is present in the deflection of the circular sheets of the first type and of the circular threads of the second and third types, viz., an asymmetry as to the terminals and as to the width of the two sheets or sets of threads. Reversing the direction of the magnetic field, or that of the current through the primary of the inductive coil, changes the position of the sheets and of their ends. Decreasing the current through the primary, or lengthening the spark-gap sufficiently, causes one sheet, or set of threads to disappear.

2. The circular sheet of the first type of spark gives the spectrum of the nitrogen bands. The central threads show that of the metallic lines and the air lines.

The second type gives the same spectrum for the bundle of central threads as for the circular threads, viz., that of the very bright air lines and the fainter metallic lines.

In the third type of spark the central threads show the same spectrum lines for each of the seven different metals which were used as spark terminals, these lines being identified with the air lines.

The spectrum of the circular threads shows the arc lines in addition to the air lines.

The spectrum of the part of the spark about the terminals shows the spark lines in addition to the arc and air lines. This gives the combined spectra of the metallic vapor, the circular and the central threads, because the metallic vapor could not be isolated.

These facts, together with certain observations presented at the end of this paper, give further evidence that the spark lines may be due to abnormal vibrations arising when the atoms are torn from the metal terminals; whereas the arc lines in the spark spectrum may be due to the more fundamental vibrations.

3. The value of the velocity of the particles associated with the circular threads is approximately 5×10^4 cm per sec. and this velocity is of the same order as that obtained for the streamers when they are measured close to the spark terminals.

These particles carry a negative charge.

They move in paths of different curvature.

Substituting in the equation

$$\rho=\frac{mv}{eH}$$

the values found for their velocity and for the curvature of their paths, $\frac{e}{m}$ is found to vary from 1.2×10^2 to 0.7×10^2.

The present investigation seems to show that in the electric discharge at atmospheric pressure there are negative particles which in a magnetic field describe luminous paths in the form of spirals and circles, similar to those described by the negative particles emitted by a hot lime cathode[1] in the discharge at low pressure.

The velocity of these particles in the discharge at atmospheric pressure is of the order of 5×10^4 cm per sec. whereas that of the particles in the discharge at low pressure is from 1.6×10^8 cm per sec. to 1.07×10^9 cm per sec.

It does not follow that these negative particles at atmospheric pressure are themselves luminous. The bright spiral and circular paths seen in a magnetic field may mean simply that the particles excite to luminescence the gas through which they pass. The nitrogen bands which constitute the spectrum of the spiral and circular sheets in the first type of spark seem to indicate either that the nega-

[1] Wehnelt, *loc. cit.*

tive particles associated with these sheets are capable of exciting a luminescence in the gas through which they pass, but have no luminescence of their own, or that the particles of air have themselves become ionized as well as excited to luminescence. The arc lines, however, which appear in addition to the air lines in the spectrum of the third type of spark, suggest that the charged particles here not only bring to luminescence the gas through which they pass but also that they themselves emit a radiation characteristic of the metal from which they appear to come.

The average velocity of the particles associated with these circular threads seems to be of the same order as that of the metallic vapor, as long as the latter is still close to the spark terminals. This agreement of the velocities and further the fact that the arc lines are present in the spectra of both the threads and the vapor, suggest some analogy between them.

Little that is definite can be said about the central threads. In the first and second types of spark they could not be deflected with a magnetic field up to 12,000, whereas in a field of this strength the central threads in the third type of spark assumed the form of spirals and semicircles, having a radius so small that measurements like those of the easily deflected threads were impossible. Thus far the spectra of these threads in the third type of spark give no clue to the nature of their mechanism.

The present investigation was suggested by Professor W. B. Huff, of Bryn Mawr College. I wish to acknowledge my indebtedness to him and to Dr. James Barnes, of Bryn Mawr College, for their helpful suggestions and criticisms during the course of the investigation.

Physical Laboratory
Bryn Mawr College
March 1908

DESCRIPTION OF PLATE

Fig. 1.—Oscillatory spark taken with the mirror in rotation. Speed of mirror —50 revolutions per sec., C=.012 mf., L=.003 henries. The lower figure shows the first discharge and six weaker discharges (*a*) which follow approximately the same path; also the short, curved streamers (*b*). The upper figure shows the trailing light (*c*). If the spark passes when the mirror is in exactly the right position, all these features may be seen in the same spark-discharge. P. 122.

Fig. 2.—Spirals. Spark-length parallel to the magnetic field. *Al* terminals. P. 127.

FIG. 3.—Circles. Spark length perpendicular to the magnetic field. The shape of the terminals did not permit complete semicircles below. P. 127.

FIG. 4.—Oscillatory spark with terminals too widely separated for the presence of the circular threads. It shows the vapor about the terminals and the irregular, central threads. P. 128.

FIG. 5.—Spark with a capacity of 0.0005 mf and no self-induction. It shows the bundle of bright threads straight across the spark-gap and the circular threads on one side of the gap. To the eye they were also present on the other side, but the short exposure of 1/50 sec. evidently "caught" the spark at an interval when they were present on one side only. Pp. 128 and 129.

FIG. 6.—Spiral form of the central threads in the oscillatory spark ($C=0.012$ mf, $L=0.003$ henries). The radius of curvature is too small for them to appear clearly in the photograph. P. 130.

FIG. 7.—To show the twofold asymmetry in the circular threads. Pp. 129, 131, and 132.

FIG. 8.—Asymmetry of sheets. Pp. 131 and 132.

FIG. 9.—Circular threads present only on one side of the spark-gap. P. 132.

FIG. 10.—Same as 8, but direction of magnetic field reversed. Pp. 131, 132, and 133.

FIG. 11.—Same as 8 but direction of current through the primary reversed. This shows the bright points of light where the sheet meets the terminals. In figs. 8, 10, 11, on the left-hand side the sheets are partly hidden by the spark terminal. Pp. 131, 132, and 133.

FIG. 12.—Lower spectrum, that of the sheet in the first type of spark, taken with the outer edge of the sheet focused upon the slit. Upper spectrum that of spark, obtained with C, and L in the circuit, taken for purposes of comparison. *Cd* terminals. These spectra are not in focus on the right owing to the plane surface of the photographic plate. P. 134.

FIG. 13.—Lower spectrum that of the central threads and sheet of the first type of spark taken close to the terminal. Upper spectrum that of the spark, obtained with C and L in circuit, taken for purposes of comparison. *Mg* terminals. P. 134.

FIG. 14.—Spectra of three different parts of the oscillatory spark ($C=0.002$ mf, $L=0.003$ henries). Upper spectrum that of the metallic vapor, central threads, and circular threads, taken close to the spark terminal. Central spectrum that of the circular threads. Lower spectrum that of the central threads, taken in the center of the spark gap with terminals so far apart that center was free from metallic vapor. *Mg* terminals. P. 136.

FIG. 15.—Spectra of central threads of oscillatory spark taken in the following order from the top of the figure: *Al, Mg, Zn, Cd, Ca, Bi.* P. 136.

FIG. 16.—To show the images of the circular threads, secured with the mirror in rotation, upon which the measurements of the velocity of the particles associated with the circular threads were made. P. 138.

FIG. 17.—Spectrum of the spark ($L=0.0015$ henries) of magnesium taken with the spark length parallel to the slit of the spectroscope. P. 143.

PLATE XIV